Regional Mathematical Olympiad-2019

Time : 3 Hours **October 20, 2019**

Instructions :

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All question carry equal marks. Maximum marks : 102**
- **Answer to each questions should start on a new page. Clearly indicate the question number.**

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1. Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

Sol. Given that x^5 & $20x + \frac{19}{x}$ are rational

So, $20x + \frac{19}{x} = t \in Q$

$\Rightarrow 20x^2 + 19 - tx = 0$

$\Rightarrow x^2 - \frac{t}{20}x + \frac{19}{20} = 0$

Let the roots of eq^n are $x_1, x_2 \in Q$

then $x_1 + x_2 = \frac{t}{20} = \alpha$

and $x_1x_2 = \frac{19}{20} = \beta$

$\Rightarrow x^2 - \alpha x + \beta = 0, \quad \alpha, \beta \in Q$

$\Rightarrow x^2 = (\alpha x - \beta)$

$\Rightarrow x^5 = x(\alpha x - \beta)^2 = x[\alpha^2 x^2 + \beta^2 - 2\alpha\beta x]$

$\Rightarrow x^5 = x[\alpha^2(\alpha x - \beta)] + \beta^2 x - 2\alpha\beta(\alpha x - \beta)$

$\Rightarrow x^5 = \alpha^3[\alpha x - \beta] - \beta x + \beta^2 x - 2\alpha^2\beta x + 2\alpha\beta^2$

$\Rightarrow x^5 = \alpha^4 x - \alpha^3\beta + 2\alpha\beta^2 + (\beta^2 - 2\alpha^2\beta - \beta)x$

It is given that x^5 is rational (let = γ)

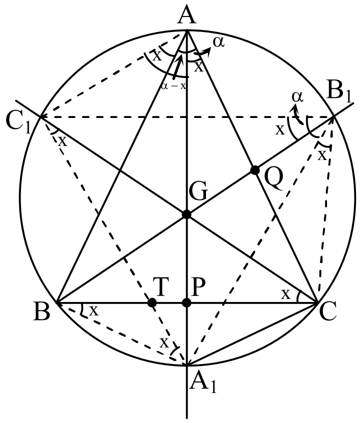
So, $(\alpha^4 + \beta^2 - 2\alpha^2\beta - \beta)x = -2\alpha\beta^2 + \alpha^3\beta + \gamma$

$x = \frac{\gamma + \alpha^3\beta - 2\alpha\beta^2}{\alpha^4 + \beta^2 - 2\alpha^2\beta - \beta}$ (Numerator & denominator are Rational)

$\therefore$ $\boxed{\text{x is also Rational}}$

2. Let ABC be a triangle with circumcircle Ω and let G be the centroid of triangle ABC. Extend AG, BG and CG to meet the circle Ω again in A_1, B_1 and C_1, respectively. Suppose $\angle BAC = \angle A_1B_1C_1$, $\angle ABC = \angle A_1C_1B_1$ and $\angle ACB = \angle B_1A_1C_1$. Prove that ABC and $A_1B_1C_1$ are equilateral triangles.

Sol.



$\angle BAC = \angle A_1B_1C_1 = \alpha$

$\angle A_1AC = x$(Assume)

$\angle A_1B_1C_1 = \angle A_1AC_1 = \alpha$ (By chord A_1C_1)

$\angle C_1AB = \angle C_1B_1B = \angle C_1CB = \angle C_1A_1B = x$ (By chord BC_1)

$\angle A_1AC = \angle A_1B_1C = \angle A_1C_1C = \angle A_1BC = x$ (Chord A_1C)

$\Rightarrow \angle A_1B \,||\, CC_1$:

Similarity $B_1C \,||\, AA_1$ & $AC_1 \,||\, BB_1$

Now : A_1 B G C is TRAPEZIUM

diagonal of Trapezium divide each other proportionality

$$\frac{PG}{PA_1} = \frac{PC}{PB}$$ {But, P is mid point of BC, Because AP is median

$\boxed{PG = PA_1}$ {So $\boxed{PB = PC}$

A_1BGC is parallelogram : $\left\{\begin{matrix} BP = PC \\ GP = A_1G \end{matrix} \quad \& BA_1 \,||\, CC_1\right\}$

Now similar $\left\{\begin{matrix} AG = A_1G \\ BG = GB_1 \\ CG = GC_1 \end{matrix}\right\}$

So G is circumcenter of ΔABC

3. Let a, b, c be positive real numbers such that a + b + c = 1. Prove that

$$\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+c^3+a^3} + \frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}$$

Sol. $$\underbrace{\frac{a}{a^2+b^3+c^3}} + \frac{b}{a^3+b^2+c^3} + \frac{c}{a^3+b^3+c^2} \quad \{a+b+c=1\}$$

$$\frac{a}{1.a^2+b^3+c^3} = \frac{a}{a^2(a+b+c)+b^3+c^3} = \frac{a}{a^3+b^3+c^3+a^2b+a^2c}$$

$$\frac{a^3+b^3+c^3+a^2b+a^2c}{5} \ge (a^7\,b^4\,c^4)^{1/5} \quad (AM \ge GM)$$

$$\frac{1}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{1}{5(a^7b^4c^4)^{1/5}}$$

$$\frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{a}{5(a^{7/5}b^{4/5}c^{4/3})} \le \frac{1}{5(a^{2/5}b^{4/5}c^{4/5})}$$

$$\frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \left(\frac{a^{3/5}.b^{1/5}.c^{1/5}}{abc}\right) \times \frac{1}{5}$$

$$\frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{(a^3bc)^{1/5}}{5(abc)}$$

$$\left\{\begin{matrix} \frac{a+a+a+b+c}{5} \ge (a^3bc)^{1/5} \\ (a^3b)^{1/5} \le \frac{3a+b+c}{5} \end{matrix}\right\}$$

$$\frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{3a+b+c}{25(abc)}$$

Similar :

$$\frac{b}{a^3+b^3+c^3+ab^2+b^2c} \le \frac{a+3b+c}{25(abc)}$$

$$\frac{c}{a^3+b^3+c^3+ac^2+bc^2} \le \frac{a+b+3c}{25(abc)}$$

add : $$\sum \frac{a}{a^3+b^3+c^3+a^2b+a^2c} \le \frac{5(a+b+c)}{25(abc)}$$

$$\le \frac{1}{5abc} \quad (\text{as } a+b+c=1)$$

Hence Proved.

4. Consider the following 3×2 array formed by using the numbers 1, 2, 3, 4, 5, 6.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

Observe that all row sums are equal, but the sum of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers 7, 8, 9, ..., 3k such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \text{ and } \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

Sol. Given $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$

$a_{11} + a_{12} = a_{21} + a_{22} = a_{31} + a_{32} = 7$

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3i}$$

and $\sum_{j=1}^{k}(a_{1j})^2 = \sum_{j=1}^{k}(a_{2j})^2 = \sum_{j=1}^{k}(a_{3j})^2$

$= 1 + 2 + 3 + 4 + ... + 3k = \frac{3k(3k+1)}{2}$

$1^2 + 2^2 + 3^2 + ... + (3k)^2 = \frac{(3k)(3k+1)(6k+1)}{6}$

$\{x_1^2 + (x_1 + 5\}^2\} - \{(x_1 + 1)^2 + (x_1 + 4)^2\} = 8$ … (i)

$\{(y_1 + 1)^2 + (y_1 + 4)^2\} - \{(y_1 + 2)^2 + (y_1 + 3)^2\} = 4$ … (ii)

$\{z_1^2 + (z_1 + 5)^2\} - \{(z_1 + 2)^2 + (z_1 + 3)^2\} = 12$ …(iii)

Also 8 + 4 = 12

$\Rightarrow x_1^2 + (x_1 + 5)^2 + (y_1 + 1)^2 + (y_1 + 4)^2 + (z_1 + 2)^2 + (z_1 + 3)^2$

$\Rightarrow (x_1 + 1)^2 + (x_1 + 4)^2 + (y_1 + 2)^2 + (y_1 + 3)^2 + z_1^2 + (z_1 + 5)^2$

$x_1 + (x_1 + 5) + (y_1 + 1) + (y_1 + 4) + (z_1 + 2) + (z_1 + 3)$

$= 2x_1 + 2y_1 + 2z_1 + 15$

$= (x_1 + 1) + (x_1 + 4) + (y_1 + 2) + (y_1 + 3) + (z_1 + z_1 + 5)$

$$\Rightarrow \begin{cases} 1^2+6^2+8^2+11^2+15^2+16^2 \\ 2^2+5^2+9^2+10^2+13^2+18^2 \\ 3^2+4^2+7^2+12^2+14^2+17^2 \end{cases} \Rightarrow \begin{pmatrix} 1 & 6 & 8 & 11 & 15 & 16 \\ 2 & 5 & 9 & 10 & 13 & 18 \\ 3 & 4 & 7 & 12 & 14 & 17 \end{pmatrix}$$

is satisfying the desired condition

similarly we can find

$$\begin{bmatrix} 1 & 6 & 8 & 11 & 18 & 13 & 21 & 23 & 25 \\ 2 & 5 & 7 & 12 & 15 & 17 & 19 & 22 & 27 \\ 3 & 4 & 9 & 10 & 14 & 16 & 20 & 24 & 26 \end{bmatrix}$$

which is satisfying all condition

Also observe

$(x_1 + 1)^2 + (x_1 + 6)^2 + (x_1 + 8)^2 + (x_1 + 11)^2 + (x_1 + 15)^2 + (x + 16)^2$

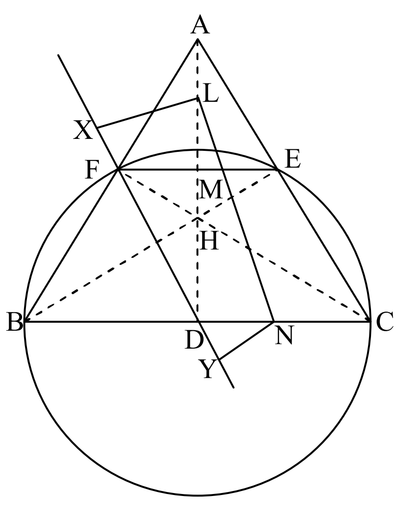
$= (x_1 + 2)^2 + (x_1 + 5)^2 + (x_1 + 9)^2 + (x_1 + 10)^2 + (x_1 + 13)^2 + (x_1 + 18)^2$

$= (x_1 + 3)^2 + (x_1 + 4)^2 + (x_1 + 7)^2 + (x_1 + 12)^2 + (x_1 + 14)^2 + (x_1 + 17)^2$

Hence we can always get a construction for k + 6 from a construction of k as we already got for k = 6 and k = 9, by induction it is true for all k such that, k > 3

5. In an acute angled triangle ABC, let H be the orthocenter, and let D, E, F be the feet of altitudes from A, B, C to the opposite sides, respectively. Let L, M, N be midpoints of segments AH, EF, BC, respectively. Let X, Y be feet of altitude from L, N on the line DF. Prove that XM is perpendicular to MY.

Sol.



$\because$ $BE \perp AC$

$CF \perp AB$

$\Rightarrow$ AFHE is cyclic quadrilateral also AH is diameter of this circle

Since L is the mid point of AH.

EF is chord of circle we have $LM \perp EF$ (as M is mid point of EF)

Similarly we have BCEF is also cyclic and N is mid point of diameter of BC

Since EF is radical axis of both circle of AFHE and BCEF and L, N are centres of the circle so $LN \perp EF$

$\Rightarrow$ L, M, N are collinear

Now LMFX, MFYN are cyclic

$\angle MLF = \angle MYF = x$ (let)

$\angle MNF = \angle MYF = y$ (let)

$\angle XMY = \angle LFN = 90^\circ$ as $x + y = 90^\circ$

$\Rightarrow \angle LFN$ is angle in the semicircle of nine point circle with LN as diameter $XM \perp MY$

6. Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that gcd (a, b) = gcd (b, c) = gcd (c, d) = gcd(d, a) = 1

Sol. Let $S \equiv \{a_i \mid 1 \le a_i \le 91, a_i \in N\}$

Maximum possible distinct pairs (a_i, a_j) such that $a_i, a_j \in S$ is ${}^{91}C_2$

Let a_i has n_i coprime numbers in S.

So a_i can form n_i pairs

Total numbers of such coprime pairs $= \frac{1}{2}\sum_{i=1}^{91} n_i$

Let $\frac{1}{2}\sum_{i=1}^{91} n_i = x$ … (1)

Now number of unordered pairs coprime with $a_1 = {}^{n_1}C_2$

Similarly, number of unordered pairs coprime with $a_2 = {}^{n_2}C_2$

So total number of such pairs is $\sum_{i=1}^{91} \frac{n_i(n_i - 1)}{2}$

These all pairs must be distinct in ordered to have 4 cyclic pairs as desired

So $\sum_{i=1}^{91} \frac{n_i(n_i - 1)}{2} \le {}^{91}C_2$

$$\sum_{i=1}^{91} \frac{n_i^2}{2} - x \le {}^{91}C_2 \qquad \text{… (2) [Using (1)]}$$

Now since $\sum_{i=1}^{91} \frac{n_i^2}{91} \ge \left(\frac{\sum n_i}{91}\right)^2$; (By weighted means concept)

So $\sum_{i=1}^{91} \frac{n_i^2}{2} \ge \left(\frac{2x}{91}\right)^2$ [Using (1)] … (3)

So from (2) & (3)

$$\frac{(2x)^2}{91} \le 2\ .\ {}^{91}C_2 + 2x$$

$$\frac{4x^2}{91} \le 90 \times 91 + 2x$$

$$16x^2 \le 360 \times 91^2 + 8 \times 91x$$

$$(4x - 91)^2 \le 91^2 \times 361$$

$$4x - 91 \le 91 \times 19$$

$$4x \le 91 \times 20$$

$$x = \frac{91 \times 20}{4}$$

$$\boxed{x \le 455}$$

Since total number of required is 455 while 456 or more such pairs are given in question

Hence 4 integers a, b, c, d can be formed such that gcd (a, b) = gcd (b, c) = gcd (c, d) = gcd (d, a) = 1.