Regional Mathematical Olympiad-2018

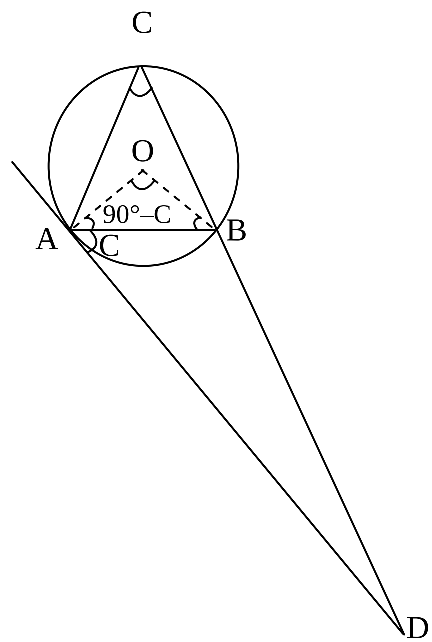
Time : 3 Hours **October 07, 2018**

Instructions :

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All question carry equal marks. Maximum marks : 102**
- **Answer to each questions should start on a new page. Clearly indicate the question number.**

1. Let ABC be a triangle with integer sides in which AB < AC. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose AD is also an integer. Prove that gcd(AB, AC) > 1.

Sol. To prove: gcd (b, c) > 1



Let AB = c

AC = b

BC = a

Given c < b

$\therefore\ \angle BAD = \angle C$

OA = OB = 90°– C

$\angle AOB = 2C$

$\therefore \angle ABD = 180° - B$

$\therefore\ \angle ADB = B - C$

In ΔABD,

$$\therefore\ \frac{c}{\sin(B - C)} = \frac{BD}{\sin C}$$

$$BD = \frac{c\sin C}{\sin B\cos C - \cos B\sin C}$$

$= \dfrac{c^2}{b\cos C - c\cos B}$

BD = $\dfrac{2c^2a}{2ab\cos C - 2ac\cos B}$

$= \dfrac{2c^2a}{(a^2+b^2-c^2)-(a^2+c^2-b^2)}$

BD = $\dfrac{ac^2}{b^2-c^2}$

Also, AD^2 = DB. DC

$= \left(\dfrac{ac^2}{b^2-c^2}\right)\left(\dfrac{ac^2}{b^2-c^2}+a\right)$

$= \dfrac{(ac^2)(ab^2)}{(b^2-c^2)^2}$

AD = $\dfrac{abc}{b^2-c^2} \in I$

$\therefore$ b & c can't have g.c.d. 1 i.e they can't be coprime as 'a' can not be divided by both (b – c) & (b + c) because b + c > a in a triangle. So b & c must be even as bc must be divided by b + c. Hence their g.c.d. > 1

2. Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n}\frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

Sol. Given : $\displaystyle\sum_{k=1}^{n}\frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}$ where $x\in R$, $n\in N$

$$\sum_{k=1}^{n}\frac{k}{x^k+\frac{1}{x^k}} = \frac{n(n+1)}{4} \qquad \ldots(1)$$

we know A.M. > G.M. if x > 0

$\therefore\ x^k + \dfrac{1}{x^k} \ge 2$

$\therefore\ \dfrac{k}{x^k+\frac{1}{x^k}} \le \dfrac{k}{2}$

$$\therefore\ \sum_{k=1}^{n}\frac{k}{x^k+\frac{1}{x^k}} \le \sum_{k=1}^{n}\frac{k}{2} = \frac{n(n+1)}{4} \qquad \ldots(2)$$

From (1) & (2) we can see that

Equation hold true if $x^k = 1 \forall\ k \in[1, n]$

$\Rightarrow$ x = 1

For x = 0, equation is not true

For x < 0, L.H.S. will be negative

$\therefore$R.H.S. $\neq$ L.H.S.

So x = 1 is only solution.

3. For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number r = 0.123123123… has period 3. If S denotes the set of all rational numbers r of the form $r = 0.\overline{abcdefgh}$ having period 8, find the sum of all the elements of S.

Sol. $r = 0.\overline{abcdefgh}$

$10^8 r = abcdefgh.\overline{abcdefgh}$

$(10^8 – 1)\, r = abcdefgh$

$r = \dfrac{abcdefgh}{10^8 - 1}$; $abcdefgh \longrightarrow 1, 2, 3, \text{--------}, 10^8 – 2$

but now numbers having period 8 also contain numbers having periods 1, 2 or 4.

Sum of numbers of period 1 = $\displaystyle\sum_{a=1}^{8} \frac{a}{10-1} = \frac{1+2+....+8}{9} = 4$

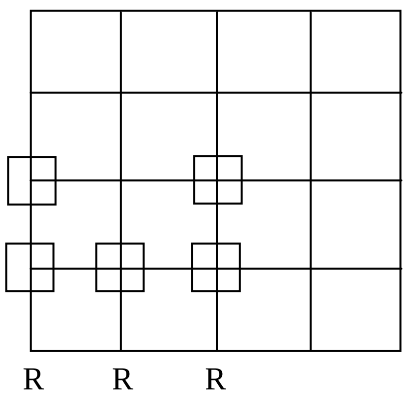
Sum of numbers of period 2 = $\displaystyle\sum_{1}^{98} \frac{ab}{10^2-1} = \frac{1+2+.....+98}{10^2-1} = 45$

Sum of numbers with period 4 = $\displaystyle\sum_{1}^{9998} \frac{abc}{10^4-1} = \frac{1+2+....+9998}{9999} = 4950$

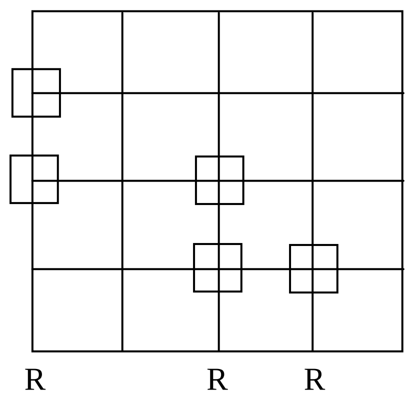
So now sum of numbers with period 8 = $\dfrac{1+2+....+99999998}{10^8-1} – 4 – 45 – 4950 = 49995000$

4. Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \le m \le 5$, $1 \le n \le 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set of the form (a, b), (a + k, b), (a + k, b + k), (a, b + k) for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square with sides parallel to axes and having at least three points of the same colour.)

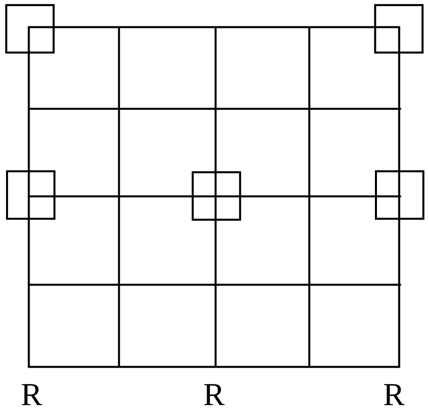
Sol. Every row has atleast 3 vertices of same colour without loss of generality we can assume that first row has 3 red then there are 4 possibilities



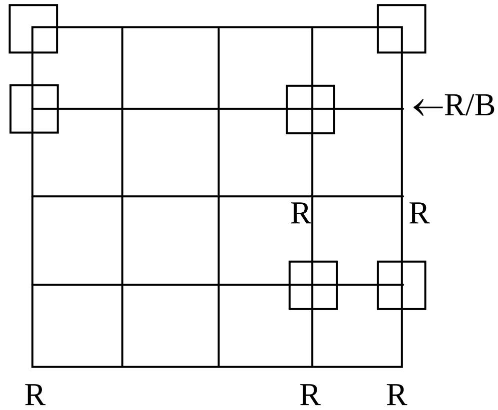
(I)



(II)



(III)



(IV)

In each figure any vertex denoted by □ can not be red colour otherwise there will be a square whose atleast 3 vertices will be blue. In fouth case vertex which is indicated by the arrow will be either red or blue. Again we get a square whose 3 vertex are of same colour.

5. Find all natural numbers n such that $1+[\sqrt{2n}]$ divides 2n. (For any real number x, [x] denotes the largest integer not exceeding x.)

Sol. Case(I) 2n is not prefect square

$x - 1 < [x] < x$, $x \in I$

$\sqrt{2n} - 1 < [\sqrt{2n}] < \sqrt{2n}$

$\sqrt{2n} < 1 + [\sqrt{2n}] < \sqrt{2n} + 1$

Let $1 + [\sqrt{2n}] = P$, $\sqrt{2n} < P < \sqrt{2n} + 1$

$P \in I$, $\quad \sqrt{2n} < P \quad\quad P < \sqrt{2n} + 1$

$\quad 2n < P^2 \quad\quad (P - 1)^2 < 2n$

$(P - 1)^2 < 2n < P^2$

$P^2 - 2P + 1 < 2n < P^2$

Now P divides $2n = P^2 - P$ only possibility

$$n = \frac{P(P-1)}{2} \ \forall\ P \geq 2, P \in I$$

Case(II) 2n is perfects square

$\sqrt{2n} = P \Rightarrow 2n = P^2$

$1 + [\sqrt{2n}] = 1 + P$ divides P^2

$\gcd(1 + P, P) = 1$ and $P \neq 0$

So $1 + P$ can not divide P^2

So no solution here

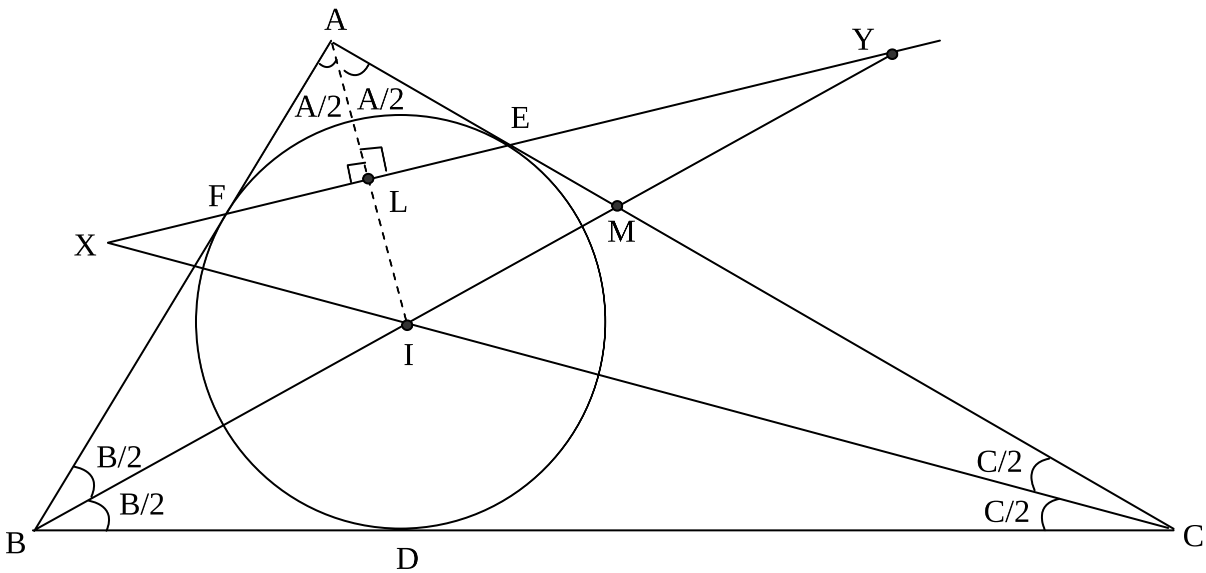
$$\therefore\ n = \frac{P(P-1)}{2} \ \forall\ P \geq 2, P \in I \text{ only solution.}$$

6. Let ABC be acute-angled triangle with AB < AC. Let I be the incentre of triangle ABC, and let D, E, F be the points at which its incircle touches the sides BC, CA, AB, respectively. Let BI, CI meet the line EF at Y, X, respectively. Further assume that both X and Y are outside the triangle ABC. Prove that

(i) B, C, Y, X are concyclic; and

(ii) I is also the incentre of triangle DYX.

Sol. **(i)**



EF is chord of incircle

So $\angle ALF = \angle ALE = 90°$

So $\angle AFL = \angle AEL = 90° - \frac{A}{2} = \frac{\pi}{2} - \frac{A}{2}$

$\angle LEA = \angle MEY$ (vertical angle)

So $\angle MEY = \frac{\pi}{2} - \frac{A}{2}$

Now, from ΔBMC

$\angle BMC = \pi - C - \frac{B}{2}$

$\angle BMC = \angle EMY$ (vertical angle)

So $\angle EMY = \pi - C - \frac{B}{2}$

from ΔEMY,

$\angle MYE = \pi - \angle EMY - \angle MEY$

$= \pi - \left(\pi - C - \frac{B}{2}\right) - \left(\frac{\pi}{2} - \frac{A}{2}\right)$

$= \pi - \pi + C + \frac{B}{2} - \frac{\pi}{2} + \frac{A}{2}$

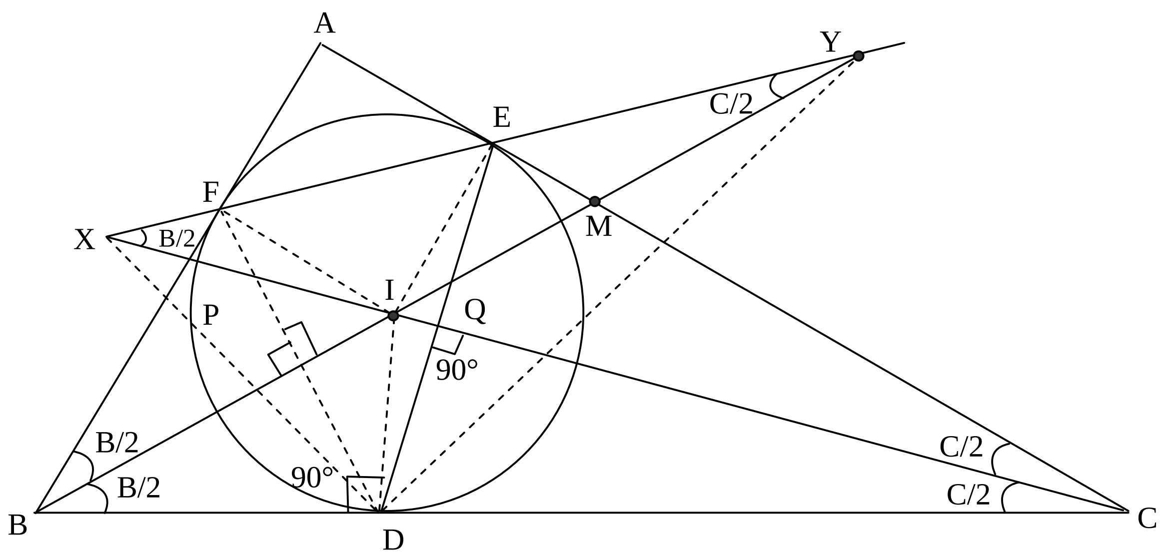
$= C + \frac{A}{2} + \frac{B}{2} - \frac{\pi}{2}$

$= C + \frac{A}{2} + \frac{B}{2} - \frac{A+B+C}{2}$

$= \frac{C}{2}$

Now chord BX subtend same angle at point Y and C. so that point B, C, Y, X are concyclic

(ii)



Point BCYX are concyclic

So $\angle XYB = \angle XCM = \frac{C}{2}$

$\angle YBC = \angle YXC = \frac{B}{2}$

ED is chord of incircle

$\angle DQC = \frac{\pi}{2}$

So $\angle QDC = \frac{\pi}{2} - \frac{C}{2}$

BC is tangent of incircle

ID is $\perp$ to BC

$\angle QDI = \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{C}{2}\right) = \frac{C}{2}$

Δ IED is isosceles Δ

$\angle IEQ = \angle IDE = \frac{C}{2}$

Clearly IECD are concyclic point

So chord DI subtend equal angle on circumference

$\angle DYI = \angle DEI = \frac{C}{2}$

So that IY is internal angle bisector of Y … (1)

Similarly:

Clearly B, F, I, D are concyclic point

$\angle BDP = \frac{\pi}{2} - \frac{B}{2}$

$\angle FDI = \frac{\pi}{2} - \angle BDP = \frac{B}{2}$

ΔPDI is isosceles triangle

$\angle FDI = \angle DFI = \frac{B}{2}$

Chord DI subtend equal angle on circumference

$\angle DFI = \angle DXI = \frac{B}{2}$

So that XI is internal angle bisector of X.

So that I is also incentre of ΔDYX.