CAREER POINT

JEE Advanced Exam 2019
(Paper & Solution)

Date : 27 / 05 / 2019

PAPER-1

PART-I (MATHEMATICS)

SECTION – 1 (Maximum Marks : 12)

- This section contains **FOUR (04)** questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

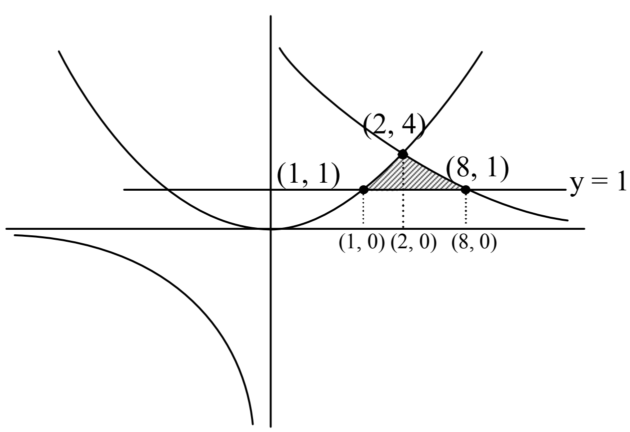
Full Marks	: **+3**	If ONLY the correct option is chosen.
Zero Marks	: **0**	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

Q.1 The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is -

(1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

Ans. **[2]**

Sol.



Required area = $\int_1^2 x^2 dx + \int_2^8 \left(\frac{8}{x}\right) dx - (7)$

$= \left[\frac{x^3}{3}\right]_1^2 + 8(\ell n x)_2^8 - 7$

$= \frac{8}{3} - \frac{1}{3} + 8\,(\ell n\,8 - \ell n\,2) - 7$

$= \frac{7}{3} + 8\,(2\,\ell n\,2) - 7$

$= -\frac{14}{3} + 16\,\ell n\,2$

Q.2 Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$, where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and I is the 2 × 2 identity matrix. If α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$, then value of $\alpha^* + \beta^*$ is -

(1) $-\frac{37}{16}$ (2) $-\frac{31}{16}$ (3) $-\frac{17}{16}$ (4) $-\frac{29}{16}$

Ans. **[4]**

Sol. $m = \sin^4\theta\cos^4\theta + (1 + \sin^2\theta)(1 + \cos^2\theta)$

$m = 2 + \sin^4\theta\cos^4\theta + \sin 2\theta\cos^2\theta$

$$\begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \frac{\beta}{|M|}\begin{bmatrix} \cos^4\theta & 1+\sin^2\theta \\ -1-\cos^2\theta & \sin^4\theta \end{bmatrix}$$

$\sin^4\theta = \alpha + \frac{\beta}{|M|}\cos^4\theta$ …(1)

$-1 - \sin^2\theta = \frac{\beta}{|M|}(1 + \sin^2\theta)$ … (2)

$1 + \cos^2\theta = \frac{\beta}{|M|}(-1 - \cos^2\theta)$ … (3)

$\cos^4\theta = \alpha + \frac{\beta}{|M|}\sin^4\theta$ …(4)

From eq^n (3) $\beta = -|M|$

$\beta = -(\sin^4\theta\cos^4\theta + \sin^2\theta\cos^2\theta + 2)$

$\beta = -(t^2 + t + 2)$ $t = \sin^2\theta\cos^2\theta$

$t = \frac{1}{4}(\sin 2\theta)^2$

$0 \le t \le \frac{1}{4}$

β_{min} at $t = \frac{1}{4}$ $\beta_{min} = -\left(\frac{1}{16} + \frac{1}{4} + 2\right)$

$\beta_{min} = -\frac{1+4+32}{16}$

$\beta_{min} = -\frac{37}{16}$

From (1)

$\sin^4\theta = \alpha - \cos^4\theta$

$\alpha = \sin^4\theta + \cos^4\theta$

$\alpha = 1 - \frac{1}{2}(\sin^2 2\theta)$

$\alpha_{min} = \frac{1}{2}$

$\alpha_{min} + \beta_{min} = \frac{-37}{16} + \frac{1}{2} = \frac{-29}{16}$ Option (4) is correct

Q.3 A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

(1) $-3 \le m < -1$ (2) $2 \le m < 4$ (3) $4 \le m < 6$ (4) $6 \le m < 8$

Ans. **[2]**

Sol. Coordinate of mid point of chord

$\left(\frac{-3}{5}, -\frac{3m}{5} + 1\right)$

Slope of line joining mid point of chord with centre (3, –2)

$= \frac{-\frac{3m}{5} + 1 + 2}{-\frac{3}{5} - 3} = \frac{-3m + 15}{-18}$

Now this line is perpendicular to given chord

$\left(\frac{-3m + 15}{-18}\right)(m) = -1$

$-3m^2 + 15m = 18$

$3m^2 - 15m + 18 = 0$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$m = 2, m = 3$

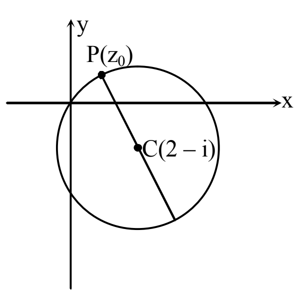
option (2) is correct $2 \le m \le 4$

Q.4 Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z - 1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is -

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) $-\frac{\pi}{2}$ (4) $\frac{3\pi}{4}$

Ans. **[3]**

Sol.



$|z-(2-i)| \geq \sqrt{5}$

for $|z_0 - 1|$ to be minimum

$z_0 = x_0 + iy_0$ is at point P as shown in figure

$\arg\left(\dfrac{4-(z_0+\bar{z}_0)}{z_0-\bar{z}_0+2i}\right) = \arg\left(\dfrac{4-2x}{2iy+2i}\right)$

$= \arg\left(\dfrac{-i(2-x)}{y+2}\right)$

$= \arg(-i\lambda) = -\dfrac{\pi}{2}$ option (3) is correct

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 marks;
  choosing ONLY (B) will get +1 marks;
  choosing ONLY (D) will get +1 marks;
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get –1 mark.

Q.1 Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$a_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct ?

(1) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Ans. **[1,2,3]**

Sol. $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2}$$

(1) $b_n = a_{n-1} + a_{n-1}$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha^2 + 1) - \beta^{n-1}(\beta^2 + 1)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha + 2) - \beta^{n-1}(\beta + 2)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}\left(\frac{5+\sqrt{5}}{2}\right) - \beta^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5}\,\alpha^{n-1}\left(\frac{\sqrt{5}+1}{2}\right) - \sqrt{5}\,\beta^{n-1}\left(\frac{\sqrt{5}-1}{2}\right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5}\,\alpha^{n-1}(\alpha) - \sqrt{5}\,\beta^{n-1}(-\beta)}{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}$$

$$b_n = \frac{\sqrt{5}\alpha^n + \sqrt{5}\beta^n}{\sqrt{5}}$$

$b_n = \alpha^n + \beta^n$ Option (1) is correct

(2) $$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{(\alpha - \beta)10^n}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left(\frac{\alpha}{10}\right)^n - \left(\frac{\beta}{10}\right)^n$$

$$= \frac{1}{\sqrt{5}} \left[\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} \right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha(10 - \beta) - \beta(10 - \alpha)}{(10 - \alpha)(10 - \beta)} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{10\alpha - \alpha\beta - 10\beta + \alpha\beta}{100 - 10(\alpha + \beta) + \alpha\beta} \right)$$

$= \frac{1}{\sqrt{5}} \frac{10(\alpha-\beta)}{100-10(\alpha+\beta)+\alpha\beta}$

$= \frac{10}{100-10-1} = \frac{10}{89}$ option (2) is correct

(3) $a_1 + a_2 + a_3 + ... + a_n$

$= \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{\alpha^i - \beta^i}{\alpha-\beta}$

$= \frac{1}{(\alpha-\beta)}\left(\frac{\alpha(1-\alpha^n)}{(1-\alpha)} - \frac{\beta(1-\beta^n)}{(1-\beta)}\right)$

$= \frac{(\alpha+1)(1-\alpha^n)-(\beta+1)(1-\beta^n)}{(1-\alpha)(1-\beta)(\alpha-\beta)}$

$= \frac{\alpha^2 - \alpha^{n+2} - \beta^2 + \beta^{n+2}}{(1-\alpha)(1-\beta)(\alpha-\beta)}$

$= \frac{\sqrt{5} + \beta^{n+2} - \alpha^{n+2}}{(\beta-\alpha)} = -1 + a_{n+2}$ option (3) is correct

(4) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n + \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n$

$= \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}} = \frac{\alpha}{10-\alpha} + \frac{\beta}{10-\beta}$

$= \frac{10(\alpha+\beta)-2\alpha\beta}{100-10(\alpha+\beta)+\alpha\beta} = \frac{10+2}{89} = \frac{12}{89}$ option (4) is wrong

Q.2 Let L_1 and L_2 denote the lines $\vec{r} = \hat{i} + \lambda(-\hat{i}+2\hat{j}+2\hat{k}),\ \lambda \in R$ and $\vec{r} = \mu(2\hat{i}-\hat{j}+2\hat{k}),\ \mu \in R$ respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(1) $\vec{r} = \frac{1}{3}(2\hat{i}+\hat{k}) + t(2\hat{i}+2\hat{j}-\hat{k}),\ t \in R$

(2) $\vec{r} = t(2\hat{i}+2\hat{j}-\hat{k}), t \in R$

(3) $\vec{r} = \frac{2}{9}(4\hat{i}+\hat{j}+\hat{k}) + t(2\hat{i}+2\hat{j}-\hat{k}),\ t \in R$

(4) $\vec{r} = \frac{2}{9}(2\hat{i}-\hat{j}+2\hat{k}) + t(2\hat{i}+2\hat{j}-\hat{k}), t \in R$

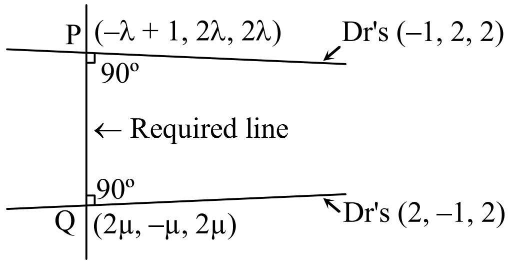
Ans. **[1,3,4]**

Sol. Line (1) : $\vec{r} = \hat{i} + \lambda(-\hat{i}+2\hat{j}+2\hat{k}),\ \lambda \in R$

Line (2) : $\vec{r} = \mu(2\hat{i}-\hat{j}+2\hat{k}),\ \mu \in R$

Line (1) $\frac{x-1}{-1} = \frac{y-0}{2} = \frac{z-0}{2} = \lambda$

Line (2) $\frac{x-0}{2} = \frac{y-0}{-1} = \frac{z-0}{2} = \mu$



Dr's of line PQ $(-\lambda - 2\mu + 1, 2\lambda + \mu, 2\lambda - 2\mu)$

Let first line parallel to $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Let second line parallel to $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(6) - \hat{j}(-2-4) + \hat{k}(1-4)$$

$$= 6\hat{i} + 6\hat{j} - 3\hat{k}$$

So that Dr's of required line (2, 2, –1)

Now $\dfrac{-\lambda - 2\mu + 1}{2} = \dfrac{2\lambda + \mu}{2} = \dfrac{2\lambda - 2\mu}{-1}$

$-\lambda - 2\mu + 1 = 2\lambda + \mu$

$3\lambda + 3\mu = 1$ ...(1)

$-2\lambda - \mu = 4\lambda - 4\mu$

$6\lambda = 3\mu$

$\mu = 2\lambda$ ...(2)

From (1) & (2)

$\lambda = \dfrac{1}{9}$

$\mu = \dfrac{2}{9}$

Point P $\left(\dfrac{8}{9}, \dfrac{2}{9}, \dfrac{2}{9}\right)$

Point Q $\left(\dfrac{4}{9}, \dfrac{-2}{9}, \dfrac{4}{9}\right)$

Now equation of required line

$\vec{r} = \left(\dfrac{8}{9}\hat{i} + \dfrac{2}{9}\hat{j} + \dfrac{2}{9}\hat{k}\right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow$ option (3) correct

$\vec{r} = \left(\dfrac{4}{9}\hat{i} - \dfrac{2}{9}\hat{j} + \dfrac{4}{9}\hat{k}\right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow$ option (4) correct

option 2 is wrong

option (1) is correct

Q.3 Let $f : R \to R$ be given by $f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$

Then which of the following options is/are correct ?

(1) f is increasing on $(-\infty, 0)$

(2) f is onto

(3) f' has a local maximum at x = 1

(4) f' is NOT differentiable at x = 1

Ans. **[2,3,4]**

Sol.

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x-2)\log_e(x-2) - x + \frac{10}{3} & x \ge 3 \end{cases}$$

$$f(x) = \begin{cases} 5(x+1)^4 - 2 & x < 0 \\ 2x - 1 & 0 \le x < 1 \\ 2x^2 - 8x + 7 & 1 \le x < 3 \\ \ell n(x-2) & x \ge 3 \end{cases}$$

$x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1$ takes value between $-\infty$ to 1

also $(x-2)\ \ell n(x-2) - x + \frac{10}{3}$ takes value between $\frac{1}{3}$ to ∞

So range of f(x) is R.

option (1) is correct

$f''(1^-) = 2$ and $f''(1^+) = -4$

so $f'(x)$ is not diff. at x = 1

so option (2) is correct

$f'(x)$ has local maxima at x = 1

so option (3) is correct

Q.4 Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ?

(1) If $M = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

(2) a + b = 3

(3) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$

(4) $\det(\text{adj } M^2) = 81$

Ans. **[1,2,3]**

Sol. $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ $\quad$ $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$2 - 3b = -1$ $\quad$ $3 - 2a = -1$

$3b = 3$ $\quad$ $-2a = -4$

$b = 1$ $\quad$ $a = 2$

Option (1)

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\beta + 2\gamma = 1$

$\alpha + 2\beta + 3\gamma = 2$

$3\alpha + \beta + \gamma = 3$

Solve it

$\alpha = 1, \beta = -1, \gamma = 1$

$\alpha - \beta + \gamma = 1 + 1 + 1 = 3$ $\quad$ option (1) correct

Option (2)

$a + b = 1 + 2 = 3$

Option (3) $(\text{adj } M)^{-1}$ $\text{adj } M^{-1}$

$= 2 \text{ adj } (M^{-1})$

$= 2\left(-\frac{M}{2}\right)$

$= -M$

Option (4)

$|\text{adj } M^2| = |M|^2 = |M|^4 = 16$

Q.5 There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

(1) Probability that the selected bag is B_3, given that the chosen ball is green, equals $\frac{5}{13}$

(2) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(3) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(4) Probability that the chosen ball is green equals $\frac{39}{80}$

Ans. [2 & 4]

Sol.

Bag I: 5G, 5R

Bag II: 5G, 3R

Bag III: 3G, 5R

Option (1) $P\left(\frac{B_3}{G}\right) = \frac{P(B_3 \cap G)}{P(G)} = \frac{\frac{4}{10}\times\frac{3}{8}}{39/80} = \frac{12}{39} = \frac{4}{13}$

Option (2) $P\left(\frac{G}{B_3}\right) = \frac{P(G \cap B_3)}{P(B_3)} = \frac{P(G)P(B_3)}{P(B_3)} = P(G) = \frac{3}{8}$

Option (3) $P(B_3 \cap G) = P(B_3)\,P(G) = \frac{4}{10} \times \frac{3}{8} = \frac{3}{20}$

Option (4) $P(G) = P(B_1)\,P\left(\frac{G}{B_1}\right) + P(B_2)\,P\left(\frac{G}{B_2}\right) + P(B_3)\,P\left(\frac{G}{B_3}\right)$

$= \frac{3}{10}\times\frac{5}{10} + \frac{3}{10}\times\frac{5}{8} + \frac{4}{10}\times\frac{3}{8}$

$= \frac{3}{20} + \frac{15}{80} + \frac{12}{80}$

$= \frac{12+15+12}{80} = \frac{39}{80}$

Option (2) & (4) correct

Q.6 Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ?

(1) $xy' + \sqrt{1-x^2} = 0$

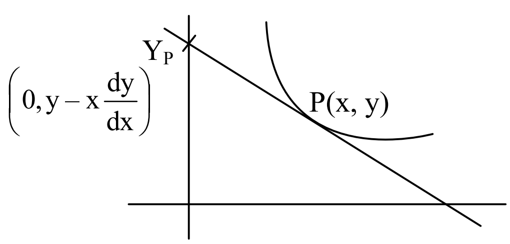
(2) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(3) $xy' - \sqrt{1-x^2} = 0$

(4) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

Ans. [1,2]

Sol.



equation of tangent

$$(Y - y) = \frac{dy}{dx}(X - x)$$

Put X = 0

$$Y = y - x.\frac{dy}{dx}$$

Distance between $PY_P = 1$

$$\sqrt{x^2 + x^2\left(\frac{dy}{dx}\right)^2} = 1$$

$$x^2 + x^2\left(\frac{dy}{dx}\right)^2 = 1$$

$$x^2\left(\frac{dy}{dx}\right)^2 = 1 - x^2$$

$$x\frac{dy}{dx} = \pm\sqrt{1-x^2}$$

$$x\frac{dy}{dx} = \sqrt{1-x^2}$$

$$\int dy = \int \frac{\sqrt{1-x^2}}{x}dx$$

put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$y = \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$y = \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$x\frac{dy}{dx} = -\sqrt{1-x^2}$$

$$dy = -\int \frac{\sqrt{1-x^2}}{x}dx$$

Integrate

$$y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + c$$

$f(1) = 0$, $c = 0$

$$y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$$

$$y = \int(\operatorname{cosec}\theta - \sin\theta)d\theta$$

$$y = \ell n(\operatorname{cosec}\theta - \cot\theta) + \cos\theta + c$$

$$y = \ell n\left(\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

$$y = \ell n\left(\frac{1-\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

$$y = \ell n\left(\frac{x^2}{1+\sqrt{1-x^2}} \times \frac{1}{x}\right) + \sqrt{1-x^2} + c$$

$$y = -\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

Now F(1) = 0

c = 0

$$y = -\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right)+\sqrt{1-x^2}$$

Curve lies in I[st] quadrant so that option (1) & (2) are correct.

Q.7 Define the collection $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1;$

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, n > 1;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, n > 1

Then which of the following options is/are correct ?

(1) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

(2) The eccentricities of E_{18} and E_{19} are NOT equal

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

Ans. **[1, 3]**

Sol. Area maximum when θ = 45º

	a	b
E_1	3	2
E_2	$\frac{3}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$
E_3	$\frac{3}{(\sqrt{2})^2}$	$\frac{2}{(\sqrt{2})^2}$

⋮

⋮

(1) $E_1 + E_2 + ... + E_m$

when $m \to \infty$ $\frac{2ab}{1-\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}} = 4ab = 4.3.2 = 24$

(2) Length of L.R. = $\frac{2b^2}{a} = \frac{2.4.24}{2.8.3} = \frac{1}{6}$

(3) Distance between focus and centre of ellipse of $I_9 = \frac{3}{2^4}.\frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{16}$

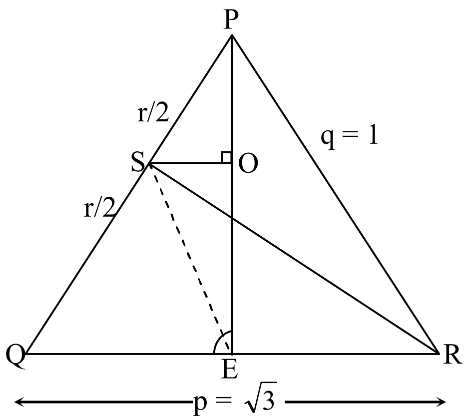
Q.8 In a non-right-angled triangle ΔPQR, let p, q, r denote the lenghts of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

(1) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$ (2) Length of $OE = \frac{1}{6}$

(3) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$ (4) Length of $RS = \frac{\sqrt{7}}{2}$

Ans. **[1,2,4]**

Sol.



$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2(1)$$

$$\Rightarrow \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2}$$

$\angle P = 60^o$ or 120^o

and $\angle Q = 30^o$ or 150^o

$\because \angle P + \angle Q < 180^o$ and $\neq 90^o$

$\angle P = 120^o$ & $\angle Q = 30^o$ & $\angle R = 30^o$

$$\frac{r}{\sin R} = 2 \Rightarrow r = 1$$

$$\text{Medians } RS = \frac{1}{2}\sqrt{2p^2+2q^2-r^2} = \frac{1}{2}\sqrt{6+2-1} = \frac{\sqrt{7}}{2}$$

$$\text{inradius} = \frac{2\Delta}{p+q+r} = \frac{\frac{2pqr}{4\times 1}}{p+q+r} = \frac{1}{2}\left(\frac{1\times 1\times\sqrt{3}}{1+1+\sqrt{3}}\right) = \frac{\sqrt{3}}{2}\left(\frac{2-\sqrt{3}}{1}\right)$$

$$\Rightarrow \frac{1}{2}\times\sqrt{3}\times PE = \frac{pqr}{4(1)} \text{ (equal area of } \Delta)$$

$$PE = \frac{1\times 1\times\sqrt{3}}{4}\times\frac{2}{\sqrt{3}} = \frac{1}{2}$$

$$\Rightarrow OE = \frac{2(\text{area of } \Delta OPQ)}{QR} = \frac{2\times\frac{1}{3}\left(\frac{1}{2}\times 1.\sqrt{3}\sin 30^o\right)}{\sqrt{3}} = \frac{1}{6}$$

SECTION – 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks : +3 If ONLY the correct numerical value is entered.

Zero Marks : 0 In all other cases.

Q.1 Three lines are given by $\vec{r} = \lambda\hat{i}$, $\lambda \in R$; $\vec{r} = \mu(\hat{i}+\hat{j})$, $\mu \in R$ and $\vec{r} = \nu(\hat{i}+\hat{j}+\hat{k})$, $\nu \in R$. Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ, then the value of $(6\Delta)^2$ equals ____

Ans. **[0.75]**

Sol. $A(\lambda, 0, 0)$

$B(\mu, \mu, 0)$

$C(\nu, \nu, \nu)$

A, B & C satisfies x + y + z = 1

$\therefore \lambda = 1$

$\mu = \frac{1}{2}$

$\nu = \frac{1}{3}$

$\therefore A(1, 0, 0), B\left(\frac{1}{2}, \frac{1}{2}, 0\right), C\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$\overrightarrow{AB} = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + 0\hat{k}$

$\overrightarrow{AC} = -\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$

$$\Delta = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| \Rightarrow \frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k} \\ -1/2 & 1/2 & 0 \\ -2/3 & 1/3 & 1/3\end{matrix}\right\|$$

$$\Delta = \frac{\sqrt{3}}{2\times 6}$$

$$\therefore 36\Delta^2 = \frac{3}{4} = 0.75$$

Q.2 If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$, then $27I^2$ equals ___

Ans. **[4.00]**

Sol. $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(2-\cos 2x)}$$

$$I = \frac{2}{\pi}\int_{0}^{\pi/4} \frac{dx}{3-2\cos^2 x}$$

$$I = \frac{2}{\pi}\int_{0}^{\pi/4} \frac{\sec^2 x}{1+3\tan^2 x}\,dx$$

$\sqrt{3}\tan x = t$

$\sec^2 x\,dx = \frac{dt}{\sqrt{3}}$

$$I = \frac{2}{\pi} \times \frac{1}{\sqrt{3}}\int_{0}^{\sqrt{3}} \frac{dt}{1+t^2}$$

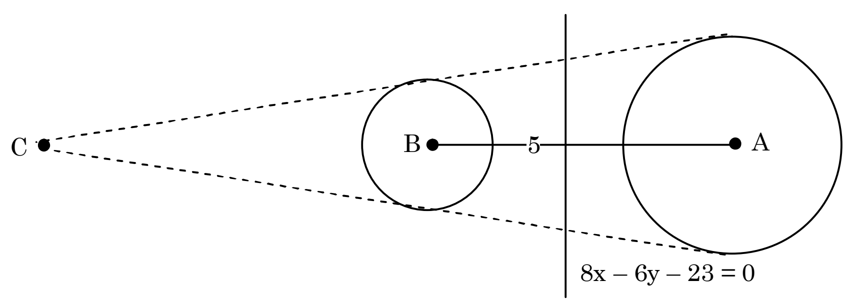
$$I = \frac{2}{\sqrt{3}\pi} \times \frac{\pi}{3} = \frac{2}{3\sqrt{3}}$$

$$27I^2 = 27 \times \frac{4}{9\times 3} = 4$$

Q.3 Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is ____.

Ans. **[10.00]**

Sol.



$$AL = \left|\frac{16-18-23}{10}\right| = \frac{5}{2}$$

$$\frac{CB}{CA} = \frac{1}{2}$$

$$\frac{CA-5}{CA} = \frac{1}{2}$$

$CA = 10$

Q.4 Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$ equals ___

Ans. **[3.00]**

Sol. $|a + b\omega + c\omega^2|^2 = a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

It will be minimum when a, b, c are consecutive integers.
So, minimum value is '3'

Q.5 Let AP(a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If $AP(1 ; 3) \cap AP(2; 5) \cap AP(3 ; 7) = AP(a; d)$ then a + d equals ____

Ans. **[157.00]**

Sol. First series {1, 4, 7, 10, 13 .... } → c.d = 3
Second series {2, 7, 12, 17 .... } → c.d = 5
Third series {3, 10, 17, 24 .... } → c.d = 7
First common term of these series is 52
New common difference of new series is
L.C.M of (3, 5, 7) = 105
∴ a = 52
d = 105
a + d = 157

Q.6 Let S be the sample space of all 3 × 3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by $E_1 = \{A \in S : \det A = 0\}$ and $E_2 = \{A \in S : \text{sum of entries of A is } 7\}$. If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

Ans. **[0.50]**

Sol. In matrix A there are 7 ones and 2 zeros
∴ Sum of elements of A = 7
Number of such matrices = ${}^9C_2$
Out of all such matrices E_1 will be those when both zeros lie in the same row or in the same column

eq. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$n(E_1 \cap E_2) = 2 \times {}^3C_2 \times {}^3C_2 = 18$

So, $n\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{16} = \frac{1}{2} = 0.50$