JEE Main Online Exam 2019

Questions & Solutions

12th April 2019 | Shift - II

MATHEMATICS

Q.1 Let a $\in \left(0, \frac{\pi}{2}\right)$ be fixed. If the integral $\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha} dx = A(x)\cos 2\alpha + B(x)\sin 2\alpha + C$, where C is a constant of integration, then the functions A(x) and B(x) are respectively :

(1) $x - \alpha$ and $\log_e|\cos(x - \alpha)|$
(2) $x + \alpha$ and $\log_e|\sin(x - \alpha)|$
(3) $x + \alpha$ and $\log_e|\sin(x + \alpha)|$
(4) $x - \alpha$ and $\log_e|\sin(x - \alpha)|$

Ans. **[4]**

Sol. $\int \frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x\cos\alpha - \cos x\sin\alpha} dx$

$= \int \frac{\sin(x+\alpha)}{\sin(x-\alpha)} dx$

$= \int \frac{\sin(x-\alpha+2\alpha)}{\sin(x-\alpha)} dx$

$= \int \frac{\sin(x-\alpha)\cos 2\alpha}{\sin(x-\alpha)} dx + \int \frac{\cos(x-\alpha)\sin 2\alpha}{\sin(x-\alpha)} dx$

$= (x - \alpha)\cos 2\alpha + \sin 2\alpha \log|\sin(x-\alpha)| + C$

Q.2 The general solution of the differential equation $(y^2 - x^3)dx - xydy = 0$ $(x \neq 0)$ is :
(where c is a constant of integration)

(1) $y^2 + 2x^2 + cx^3 = 0$
(2) $y^2 + 2x^3 + cx^2 = 0$
(3) $y^2 - 2x + cx^3 = 0$
(4) $y^2 - 2x^3 + cx^2 = 0$

Ans. **[2]**

Sol. $(y^2 - x^3)dx - xydy = 0 \quad (x \neq 0)$

$y^2 - x^3 - xy\frac{dy}{dx} = 0$

$xy\frac{dy}{dx} - y^2 = -x^3$

$y\frac{dy}{dx} - \frac{1}{x}y^2 = -x^2 \qquad \ldots(i)$

Let $y^2 = v \qquad 2y\frac{dy}{dx} = \frac{dv}{dx}$

put in eq^n (i)

$\frac{1}{2}\frac{dv}{dx} - \frac{1}{x}v = -x^2$

$\frac{dv}{dx} + \left(-\frac{2}{x}\right)v = -2x^2 \qquad \ldots(ii)$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\ell n x} = \frac{1}{x^2}$$

solution of eq^n (ii)

$$v \times \frac{1}{x^2} = \int -2x^2 \times \frac{1}{x^2}dx \; - C$$

$$\frac{v}{x^2} = -2x - C$$

$$y^2 = -2x^3 - cx^2$$

$$y^2 + 2x^3 + cx^2 = 0$$

Q.3 The equation of common tangent to the curves $y^2 = 16x$ and $xy = -4$, is :

(1) $x - y + 4 = 0$ (2) $x + y + 4 = 0$ (3) $x - 2y + 16 = 0$ (4) $2x - y + 2 = 0$

Ans. **[1]**

Sol. $y = mx + \frac{4}{m}$ is always tangent to $y^2 = 16x$ … (i)

if it is tangent to the $xy = -4$

$$x\left(mx + \frac{4}{m}\right) = -4$$

$$m^2x^2 + 4x = -4m$$

$$m^2x^2 + 4x + 4m = 0$$

for tangent D = 0

$16 - 16m^3 = 0 \Rightarrow m = 1$ put in eq^n (i)

$$y = x + 4$$

Q.4 A plane which bisects the angle between the two given planes $2x - y + 2z - 4 = 0$ and $x + 2y + 2z - 2 = 0$, passes through the point :

(1) (1, –4, 1) (2) (1, 4, –1) (3) (2, 4, 1) (4) (2, –4, 1)

Ans. **[4]**

Sol. Eq^n of angle bisectors are

$$\frac{2x - y + 2z - 4}{\sqrt{2^2 + (-1)^2 + 2^2}} = \pm\left(\frac{x + 2y + 2z - 2}{\sqrt{1^2 + 2^2 + z^2}}\right) \quad \text{… (i)}$$

Case I : take positive sign

$$2x - y + 2z - 4 = x + 2y + 2z - 2$$

$$x - 3y - 2 = 0 \quad \text{… (ii)}$$

Case-II : take negative sign

$$2x - y + 2z - 4 = -(x + 2y + 2z - 2)$$

$$2x - y + 2z - 4 = -x - 2y - 2z + 2$$

$$3x + y + 4z - 6 = 0 \quad \text{… (iii)}$$

option (4) satisfy eq^n (iii)

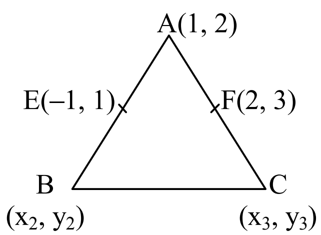
$\Rightarrow$ (2, –4, 1)

Q.5 A triangle has a vertex at (1, 2) and the mid points of the two sides through it are (–1, 1) and (2, 3). Then the centroid of this triangle is :

(1) $\left(\frac{1}{3}, 2\right)$ (2) $\left(\frac{1}{3}, \frac{5}{3}\right)$ (3) $\left(1, \frac{7}{3}\right)$ (4) $\left(\frac{1}{3}, 1\right)$

Ans. **[1]**

Sol.



$\frac{x_2+1}{2} = -1, \quad \frac{y_2+2}{2} = 1$ | $\frac{x_3+1}{2} = 2 \ \& \ \frac{y_3+2}{2} = 3$

$x_2 = -3, y_2 = 0$ | $x_3 = 3, y_3 = 4$

B(–3, 0) | C(3, 4)

centroid $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$$\left(\frac{1-3+3}{3}, \frac{2+0+4}{3}\right) = \left(\frac{1}{3}, 2\right)$$

Q.6 The Boolean expression $\sim(p \Rightarrow (\sim q))$ is equivalent to :

(1) $p \wedge q$ (2) $(\sim p) \Rightarrow q$ (3) $q \Rightarrow \sim p$ (4) $p \vee q$

Ans. **[1]**

Sol. $-(p \rightarrow \sim q) = p \wedge q$

Q.7 Let A, B and C be sets such that $\phi \neq A \cap B \subseteq C$. Then which of the following statements is not true ?

(1) If $(A - B) \subseteq C$, then $A \subseteq C$ (2) $B \cap C \neq \phi$

(3) $(C \cup A) \cap (C \cup B) = C$ (4) If $(A - C) \subseteq B$, then $A \subseteq B$

Ans. **[4]**

Sol. Let A = {1, 2, 3, 4} B = {3, 4, 5, 6} C = {1, 2, 3, 4, 7, 8}

Here $A \cap B = \{3, 4\} \subseteq C$

$A - C = \phi \subseteq B$

but $A \not\subseteq B$

So not true (wrong) statement is 4^{th}

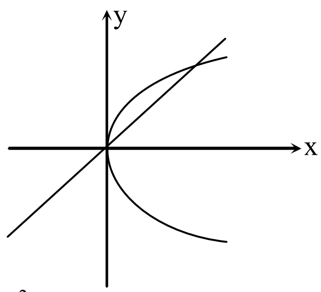
If $A - C \subseteq B$ then $A \subseteq B$

Q.8 If the area (in sq. units) bounded by the parabola $y^2 = 4\lambda x$ and the line $y = \lambda x$, $\lambda > 0$, is $\frac{1}{9}$, then λ is equal to :

(1) $4\sqrt{3}$ (2) $2\sqrt{6}$ (3) 48 (4) 24

Ans. **[4]**

Sol.



$y^2 = 4\lambda x$ & $y = \lambda x$

$\lambda^2x^2 = 4\lambda x$

$x = 0$ & $x = \frac{4}{\lambda}$

Area = $\int_0^{4/\lambda}(\sqrt{4\lambda x} - \lambda x)\,dx = \frac{1}{9}$

$\Rightarrow 2\sqrt{\lambda} \times \left(\frac{x^{3/2}}{\frac{3}{2}}\right)_0^{4/\lambda} - \lambda\left(\frac{x^2}{2}\right)_0^{4/\lambda} = \frac{1}{9}$

$\frac{4}{3}\sqrt{\lambda} \times \frac{(2^2)^{3/2} x}{\lambda^{3/2}} - \frac{\lambda}{2} \times \frac{16}{\lambda^2} = \frac{1}{9}$

$\Rightarrow \frac{32}{3\lambda} - \frac{8}{\lambda} = \frac{1}{9}$

$\Rightarrow \frac{8}{3\lambda} = \frac{1}{9}$ $\Rightarrow \lambda = 24$

Q.9 If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to :

(1) $\alpha\gamma$ (2) 0 (3) $\alpha\beta$ (4) $\beta\gamma$

Ans. **[4]**

Sol. α, β, γ are in G.P.

$\alpha x^2 + 2\beta x + \gamma = 0$ & $x^2 + x - 1 = 0$ have a common roots. Both roots will be common

$\frac{\alpha}{1} = \frac{2\beta}{1} = \frac{\gamma}{-1} = \lambda$

$\alpha = \lambda$, $B = \frac{\lambda}{2}$, $\gamma = -\lambda$

$\alpha(\beta + \gamma) = \lambda\left(\frac{\lambda}{2} - \lambda\right) = \frac{-\lambda^2}{2} = \beta\gamma$

Q.10 If $^{20}C_1 + (2^2)\ ^{20}C_2 + (3^2)\ ^{20}C_3 + ... + (20^2)^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to :

(1) (420, 19) (2) (420, 18) (3) (380, 18) (4) (380, 19)

Ans. **[2]**

Sol. $(1 + x)^{20} = {}^{20}C_0 + {}^{20}C_1x + {}^{20}C_2x^2 + ... + {}^{20}C_{20}x^{20}$ … (i)

different eq^n (i) w.r.t. x

$20(1 + x)^{19} = {}^{20}C_1.1 + 2.{}^{20}C_2x + ... + 20{}^{20}C_{20}x^{19}$ … (ii)

Multiply eq^n (ii) by x

$20x(1 + x)^{19} = {}^{20}C_1.x + 2.{}^{20}C_2x^2 + ... + 20{}^{20}C_{20}x^{20}$ … (iii)

diff. eq^n (iii) w.r.t. x

$20[(1 + x)^{19} + 19x(1 + x)^{18}] = 1.{}^{20}C_1 + 2^2.{}^{20}C_2x + ... + (20^2)^{20}C_{20}x^{19}$ … (iv)

put x = 1 in eq^n (iv)

$20(2^{19} + 19.2^{18}) = 1^2.{}^{20}C_1 + 2^2.{}^{20}C_2 + ... + (20^2)\ {}^{20}C_{20}$

$= 20 \times 2^{18}(2 + 19) = 20 \times 21 \times 2^{18} = 420 \times 2^{18}$

$A = 420, \beta = 18$

Q.11 The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines

$\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} + \hat{j}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$ is :

(1) $\frac{1}{3}$ (2) $\frac{1}{\sqrt{3}}$ (3) 3 (4) $\sqrt{3}$

Ans. **[4]**

Sol. Equation of plane containing both lines is

$$\begin{vmatrix} x-1 & y-1 & z \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix} = 0$$

$(x - 1)(-4 + 1) + (y - 1)(1 + 2) + z(1 + 2) = 0$

$-3(x - 1) + 3(y - 1) + 3z = 0$

$-x + 1 + y - 1 + z = 0$

$-x + y + z = 0$ distance from point (2, 1, 4) is

$$\left|\frac{-2+1+4}{\sqrt{1^2+1^2+1^2}}\right| = \sqrt{3}$$

Q.12 A circle touching the x-axis at (3, 0) and making an intercept of length 8 on the y-axis passes through the point :

(1) (1, 5) (2) (2, 3) (3) (3, 5) (4) (3, 10)

Ans. **[4]**

Sol. Equation of required circle will be

$(x - 3)^2 + (y \pm r)^2 = r^2$

$x^2 - 6x + 9 + y^2 \pm 2ry + r^2 = r^2$

$x^2 + y^2 - 6x \pm 2ry + 9 = 0$ …(i)

Length of y intercept $= 2\sqrt{f^2 - c}$ $f = \pm r$

$8 = 2\sqrt{r^2 - 9}$

$16 = r^2 - 9$

$r = 5$

So eq^n of required circle will be

$$x^2 + y^2 - 6x \pm 10y + 9 = 0$$

two circles

$x^2 + y^2 - 6x + 10y + 9 = 0$ … (ii) $\quad$ $x^2 + y^2 - 6x - 10y + 9 = 0$ … (iii)

option 4th (3, 10) satisfy eq^n (iii)

Q.13 A person throws two fair dice. He wins Rs. 15 for throwing a doublet (same numbers on the two dice), wins Rs. 12 when the throw results in the sum of 9, and loses Rs. 6 for any other outcome on the throw. Then the expected gain/loss (in Rs.) of the person is :

(1) $\frac{1}{4}$ loss $\quad$ (2) $\frac{1}{2}$ gain $\quad$ (3) $\frac{1}{2}$ loss $\quad$ (4) 2 gain

Ans. **[3]**

Sol.

win	+15	+12	–6
Prob.	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{26}{36}$

Probability of doublet = $\frac{6}{36}$

Probability of sum of 9 = $\frac{4}{36}$

Other probability = $\frac{26}{36}$

Expected gain/loss $= 15 \times \frac{6}{36} + 12 \times \frac{4}{36} - 6 \times \frac{26}{36}$

$$= \frac{90}{36} + \frac{48}{36} - \frac{156}{36} = \frac{-1}{2} \Rightarrow \frac{-1}{2}$$

So, $\frac{1}{2}$ loss

Q.14 Let $\alpha \in R$ and the three vectors $\vec{a} = \alpha\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \alpha\hat{k}$ and $\vec{c} = \alpha\hat{i} - 2\hat{j} + 3\hat{k}$. Then the set $S = \{\alpha : \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar$\}$

(1) contains exactly two numbers only one of which is positive

(2) is singleton

(3) contains exactly two positive numbers

(4) is empty

Ans. **[4]**

Sol. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix} = 0$$

$\alpha(3 - 2\alpha) + 1(-\alpha^2 - 6) + 3(-4 - \alpha) = 0$

$3\alpha - 2\alpha^2 - \alpha^2 - 6 - 12 - 3\alpha = 0$

$-3\alpha^2 - 18 = 0$

$\alpha^2 + 6 = 0$ $\quad$ not possible for real α

S is empty set

Q.15 The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point :

(1) $\left(\frac{5}{2}, -1\right)$ (2) $\left(-\frac{5}{2}, -1\right)$ (3) $\left(\frac{5}{2}, 1\right)$ (4) $\left(-\frac{5}{2}, 1\right)$

Ans. **[1]**

Sol. $x - y - 3 = 0$ … (i) will be chord of contact of parabola

$y = x^2 - 4x + 3$

Let the required point is $P(x_1, y_1)$ chord of contact for point P is

$$\frac{y + y_1}{2} = xx_1 - 4\frac{(x + x_1)}{2} + 3$$

$y + y_1 = 2x_1x - 4x - 4x_1 + 6$

$(2x_1 - 4)x - y + (-4x_1 - y_1 + 6) = 0$ … (ii)

eq^n (i) & (ii) are same line

$$\frac{2x_1 - 4}{1} = \frac{-1}{-1} = \frac{-4x_1 - y_1 + 6}{-3}$$

$\Rightarrow 2x_1 - 4 = 1$ | $-4x_1 - y_1 + 6 = -3$

$x_1 = \frac{5}{2}$ | $-10 - y_1 + 9 = 0$

$y_1 = -1$

Ans. $\left(\frac{5}{2}, -1\right)$

Q.16 If [x] denotes the greatest integer $\le x$, then the system of linear equations $[\sin\theta]x + [-\cos\theta]y = 0$, $[\cot\theta]x + y = 0$

(1) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(2) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(3) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(4) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

Ans. **[2]**

Sol. $[\sin\theta]x + [-\cos\theta]y = 0$ … (i)

$[\cot\theta]x + y = 0$ … (ii)

Case-I :

When $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ $\sin\theta \in \left(\frac{\sqrt{3}}{2}, 1\right)$

$\cos\theta \in \left(-\frac{1}{2}, 0\right)$ $-\cos\theta \in \left(0, \frac{1}{2}\right)$

$\cot\theta \in \left(\frac{-1}{\sqrt{3}}, 0\right)$

$[\sin\theta] = 0$, $[-\cos\theta] = 0$, $[\cot\theta] = -1$

eq^n (i) & (ii) will

$\left.\begin{matrix} 0x + 0y = 0 \\ -x + y = 0 \end{matrix}\right]$ system will have infinitely many solution

Case-II :

When $\theta \in \left(\pi, \frac{7\pi}{6}\right)$ $\sin\theta \in \left(-\frac{1}{2}, 0\right)$

$\cos\theta \in \left(-1, \frac{-\sqrt{3}}{2}\right)$

$\cot\theta \in (\sqrt{3}, \infty)$

$[\sin\theta] = -1$, $[\cos\theta] = -1$

$[\cot\theta] = \{1,2,3,.....\}$

$-x - y = 0$

$Ix + y = 0$ $\quad\quad$ $I = \{1, 2, ...\}$

It will have unique solution is all cases x = 0, y = 0

Q.17 A group of students comprises of 5 boys and n girls. If the number of ways, in which a team of 3 students can randomly be selected from this group such that there is at least one boy and at least one girl in each team, is 1750, then n is equal to :

(1) 24 (2) 25 (3) 27 (4) 28

Ans. **[2]**

Sol. Given 5 boys and n girls

total ways of forming team of 3 member under given condition

$= {}^5C_1 . {}^nC_2 + {}^5C_2 . {}^nC_1$

$\Rightarrow {}^5C_1 . {}^nC_2 + {}^5C_2 . {}^nC_1 = 1750$

$\Rightarrow \frac{5n(n-1)}{2} + 10n = 1750$

$\Rightarrow \frac{n(n-1)}{2} + 2n = 350$

$\Rightarrow n^2 + 3n = 700$

$\Rightarrow n^2 + 3n - 700 = 0$

$\Rightarrow n = 25$

Q.18 An ellipse, with foci at (0, 2) and (0, –2) and minor axis of length 4, passes through which of the following points ?

(1) $(2, \sqrt{2})$ (2) $(2, 2\sqrt{2})$ (3) $(\sqrt{2}, 2)$ (4) $(1, 2\sqrt{2})$

Ans. **[3]**

Sol. Given 2a = 4 and 2be = 4

$\Rightarrow a = 2, be = 2$

$\Rightarrow b^2e^2 = 4$

$\Rightarrow b^2 - a^2 = 4$

$\Rightarrow b^2 = 8$

$\Rightarrow$ equation of ellipse

$$\frac{x^2}{4} + \frac{y^2}{8} = 1$$

Clearly option (3) satisfy the given curve.

Q.19 For an initial screening of an admission test, a candidate is given fifty problems to solve. If the probability that the candidate solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problems is :

(1) $\frac{164}{25}\left(\frac{1}{5}\right)^{48}$ (2) $\frac{316}{25}\left(\frac{4}{5}\right)^{48}$ (3) $\frac{201}{5}\left(\frac{1}{5}\right)^{49}$ (4) $\frac{54}{5}\left(\frac{4}{5}\right)^{49}$

Ans. **[4]**

Sol. Total problems = 50

P(Solving) = $\frac{4}{5}$

P(Not solving) = $\frac{1}{5}$

P(unable to solve less than two problems)

= P(not solving one problem) + P(not solving zero problem)

$= {}^{50}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{50} + {}^{50}C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^{49}$

$= \frac{4^{50}}{5^{50}} + 50.\frac{4^{49}}{5.5^{49}}$

$= \left(\frac{4}{5}\right)^{50} + 10.\left(\frac{4}{5}\right)^{49}$

$= \left(\frac{4}{5}\right)^{49} + \left(\frac{4}{5}+10\right)$

$= \frac{54}{5}.\left(\frac{4}{5}\right)^{49}$

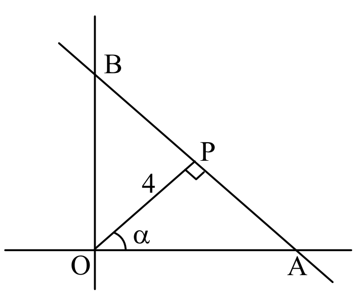
Q.20 A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60º with the line x + y = 0. Then an equation of the line L is :

(1) $x + \sqrt{3}y = 8$ (2) $\sqrt{3}x + y = 8$

(3) $(\sqrt{3}+1)x + (\sqrt{3}-1)y = 8\sqrt{2}$ (4) $(\sqrt{3}-1)x + (\sqrt{3}+1)y = 8\sqrt{2}$

Ans. **[3,4]**

Sol.



OP = 4

Given OP makes 60º with x + y = 0

let slope of OP = m

$\Rightarrow \tan 60^o = \left|\frac{m+1}{1-m}\right|$

$\Rightarrow \frac{m+1}{m-1} = \sqrt{3}$ or $-\sqrt{3}$

$\Rightarrow m + 1 = \sqrt{3}m - \sqrt{3}$ or $m + 1 = \sqrt{3} - \sqrt{3}\,m$

$\Rightarrow m(\sqrt{3} - 1) = \sqrt{3} + 1$ or $m(1 + \sqrt{3}) = \sqrt{3} - 1$

$\Rightarrow m = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $m = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\Rightarrow \tan\alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\Rightarrow$ eq^n of line $x\cos\alpha + y\sin\alpha = P$

$\Rightarrow (\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 8\sqrt{2}$ or $(\sqrt{3} - 1)x + (\sqrt{3} + 1)y = 8\sqrt{2}$

Q.21 A value of α such that $\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right)$ is :

(1) 2 (2) –2 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

Ans. **[2]**

Sol. $\int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e\left(\frac{9}{8}\right)$

$\Rightarrow \int_{\alpha}^{\alpha+1} \frac{(x+\alpha+1)-(x+\alpha)}{(x+\alpha)(x+\alpha+1)}dx = \log_e\left(\frac{9}{8}\right)$

$\Rightarrow \int_{\alpha}^{\alpha+1} \frac{dx}{x+\alpha} - \int_{\alpha}^{\alpha+1} \frac{dx}{x+\alpha+1} = \log_e\left(\frac{9}{8}\right)$

$\Rightarrow \log_e\left(\frac{x+\alpha}{x+\alpha+1}\right)\Big|_{\alpha}^{\alpha+1} = \log_e\left(\frac{9}{8}\right)$

$\Rightarrow \log_e\left(\frac{2\alpha+1}{2\alpha+2}\right) - \log\left(\frac{2\alpha}{2\alpha+1}\right) = \log_e\left(\frac{9}{8}\right)$

$\Rightarrow \log\left[\left(\frac{2\alpha+1}{2\alpha+2}\right)\left(\frac{2\alpha+1}{2\alpha}\right)\right] = \log_e\frac{9}{8}$

$\Rightarrow \frac{(2\alpha+1)^2}{4\alpha(\alpha+1)} = \frac{9}{8}$

$\Rightarrow 8[4\alpha^2 + 4\alpha + 1] = 9[4\alpha^2 + 4\alpha]$

$\Rightarrow 32\alpha^2 + 32\alpha + 8 = 36\alpha^2 + 36\alpha$

$\Rightarrow 4\alpha^2 + 4\alpha - 8 = 0$

$\Rightarrow \alpha^2 + \alpha - 2 = 0$

$\Rightarrow (\alpha + 2)(\alpha - 1) = 0$

$\Rightarrow \alpha = 1, -2$

Q.22 $\lim_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$ is :

(1) 6 (2) 1 (3) 3 (4) 2

Ans. **[4]**

Sol. $\lim_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$

$= \lim_{x\to 0}\dfrac{x+2\sin x}{x^2+2\sin x+1-\sin^2 x+x-1}\times(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1})$

$= \lim_{x\to 0}\dfrac{x+2\sin x}{x^2+2\sin x-\sin^2 x+x}\times(2)$

Applying L'H Rule

$= \lim_{x\to 0}\dfrac{2(1+2\cos x)}{2x+2\cos x-2\sin x\cos x+1} = \dfrac{2(3)}{2+1} = 2$

Q.23 The derivative of $\tan^{-1}\left(\dfrac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\dfrac{x}{2}$, where $\left(x\in\left(0,\dfrac{\pi}{2}\right)\right)$ is :

(1) $\dfrac{2}{3}$ (2) 1 (3) 2 (4) $\dfrac{1}{2}$

Ans. **[3]**

Sol. Given $y=\tan^{-1}\left(\dfrac{\sin x-\cos x}{\sin x+\cos x}\right)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{\tan x-1}{\tan x+1}\right)$

$\Rightarrow y=-\tan^{-1}\left(\dfrac{1-\tan x}{1+\tan x}\right)$

$\Rightarrow y=-\tan^{-1}\left[\tan\left(\dfrac{\pi}{4}-x\right)\right]$

$\because\ 0<x<\dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{2}<-x<0$

$\Rightarrow -\dfrac{\pi}{4}<\dfrac{\pi}{4}-x<0$

$\Rightarrow y=-\left(\dfrac{\pi}{4}-x\right)$ $\left\{\because \tan^{-1}\tan x = x\ \forall\ x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)\right.$

$\Rightarrow y=x-\dfrac{\pi}{4}$

$\dfrac{dy}{d(x/2)}=\dfrac{1}{(1/2)}=2$

Q.24 Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha \sin x = 2\alpha - 7$ has a solution. Then S is equal to :

(1) [2, 6] (2) [3, 7] (3) [1, 4] (4) R

Ans. **[1]**

Sol. Given $\cos 2x + \alpha \sin x = 2\alpha - 7$

$\Rightarrow 1 - 2\sin^2 x + \alpha \sin x = 2\alpha - 7$

$\Rightarrow 2\sin^2 x - \alpha \sin x + 2\alpha - 8 = 0$

$\Rightarrow \sin x = \dfrac{\alpha \pm \sqrt{\alpha^2 - 8(2\alpha - 8)}}{4}$

$\Rightarrow \sin x = \dfrac{\alpha \pm (\alpha - 8)}{4}$

$\Rightarrow \sin x = \dfrac{\alpha + \alpha - 8}{4}, \dfrac{\alpha - \alpha + 8}{4}$

$\sin x = 2$ (Not possible)

for solution

$-1 \le \dfrac{2\alpha - 8}{4} \le 1$

$-4 \le 2a - 8 \le 4$

$\Rightarrow 4 \le 2\alpha \le 12$

$\Rightarrow \alpha \in [2, 6]$

Q.25 Let $z \in C$ with $\text{Im}(z) = 10$ and it satisfies $\dfrac{2z - n}{2z + n} = 2i - 1$ for some natural number n. Then :

(1) n = 20 and Re(z) = –10 (2) n = 40 and Re(z) = 10

(3) n = 40 and Re(z) = –10 (4) n = 20 and Re(z) = 10

Ans. **[3]**

Sol. Let $z = x + 10i$

given $\dfrac{2z - n}{2z + n} = 2i - 1$

$\Rightarrow \dfrac{2(x + 10i) - n}{2(x + 10i) + n} = 2i - 1$

$\Rightarrow (2x - n) + 20i = (2i - 1)\,[(2x + n) + 20i]$

Comparing real and imaginary part

$\Rightarrow 2x - n = 2(-20) - (2x + n)$ and $20 = 2(2x + n) - 20$

$\Rightarrow 2x - n = -40 - 2x - n$ and $20 = 4x + 2n - 20$

$\Rightarrow 4x = -40$ and $4x + 2n = 40$

$\Rightarrow x = -10$ and $-40 + 2n = 40$

$\Rightarrow n = +40$

$\Rightarrow n = 40$ and Re(z) = –10

Q.26 Let $f(x) = 5 - |x - 2|$ and $g(x) = |x + 1|$, $x \in R$. If f(x) attains minimum value at β, then $\lim\limits_{x \to -\alpha\beta} \dfrac{(x - 1)(x^2 - 5x + 6)}{x^2 - 6x + 8}$ is equal to :

(1) $\dfrac{1}{2}$ (2) $-\dfrac{1}{2}$ (3) $\dfrac{3}{2}$ (4) $-\dfrac{3}{2}$

Ans. [1]

Sol. $f(x) = 5 - |x - 2|$

f(x) attains maximum value when

$|x - 2| = 0 \Rightarrow x = 2 = \alpha$

$g(x) = |x + 1|$

g(x) attains minimum value of $x = -1 = \beta$

$$\lim_{x \to -\alpha\beta} \frac{(x-1)(x^2-5x+6)}{x^2-6x+8}$$

$$= \lim_{x \to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)}$$

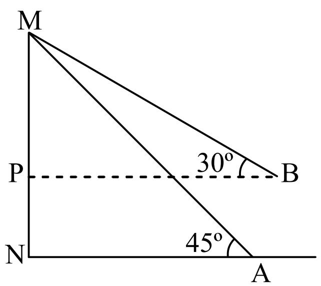
$$= \frac{(2-1)(2-3)}{(2-4)} = \frac{1}{2}$$

Q.27 The angle of elevation of the top of a vertical tower standing on a horizontal plane is observed to be 45º from a point A on the plane. Let B be the point 30 m vertically above the point A. If the angle of elevation of the top of the tower from B be 30º, then the distance (in m) of the foot of the tower from the point A is :

(1) $15(1+\sqrt{3})$ (2) $15(3-\sqrt{3})$ (3) $15(3+\sqrt{3})$ (4) $15(5-\sqrt{3})$

Ans. [3]

Sol.



AB = 30 m = NP

In ΔANM

$$\tan 45^\circ = \frac{MN}{AN} = 1$$

$\Rightarrow MN = AN$

$PM = MN - 30$

$\quad = AN - 30$

In ΔBPM

$$\tan 30^\circ = \frac{PM}{PB} = \frac{AN-30}{AN}$$

$$\frac{1}{\sqrt{3}} = \frac{AN-30}{AN}$$

$$AN = \sqrt{3}\,AN - 30\sqrt{3}$$

$$AN = \frac{30\sqrt{3}}{\sqrt{3}-1} = \frac{30\sqrt{3}(\sqrt{3}+1)}{2} = 15\,(3+\sqrt{3})$$

Q.28 A value of $\theta \in \left(0, \frac{\pi}{3}\right)$, for which $\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$, is :

(1) $\frac{\pi}{18}$ (2) $\frac{\pi}{9}$ (3) $\frac{7\pi}{24}$ (4) $\frac{7\pi}{36}$

Ans. **[2]**

Sol. $\theta \in \left(0, \frac{\pi}{3}\right)$

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2$

$$\Rightarrow \begin{vmatrix} 2 & \sin^2\theta & 4\cos 6\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Expanding along first column

$\Rightarrow 2[1 - 0] - 1[-4\cos 6\theta] = 0$

$\Rightarrow 2 + 4\cos 6\theta = 0$

$\Rightarrow \cos 6\theta = -\frac{1}{2}$

$\Rightarrow 6\theta = \frac{2\pi}{3} \Rightarrow \theta = \frac{\pi}{9}$

Q.29 If $a_1, a_2, a_3, ...$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is :

(1) 120 (2) 150 (3) 280 (4) 200

Ans. **[4]**

Sol. $a_1, a_2, a_3, ..., a_n$ are in A.P.

$a_1 + a_7 + a_{16} = 40$

$\Rightarrow a + a + 6d + a + 15d = 40$

$\Rightarrow 3a + 21d = 40$

$\Rightarrow a + 7d = \frac{40}{3}$

$S_{15} = \frac{15}{2}[2a + 14d]$

$= 15[a + 7d]$

$= 15 \times \frac{40}{3}$

$= 200$

Q.30 The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right).\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to :

(1) –36 (2) –108 (3) 36 (4) –72

Ans. **[1]**

Sol. $\left(\frac{1}{60}-\frac{x^8}{81}\right).\left(2x^2-\frac{3}{x^2}\right)^6$

term independent of x will be

$\frac{1}{60}\times$ term independent of x in $\left(2x^2-\frac{3}{x^2}\right)^6-\frac{1}{8}\times$ term of z^{-8} in $\left(2x^2-\frac{3}{x^2}\right)^6$

T_{r+1} in $\left(2x^2-\frac{3}{x^2}\right)^6$ will be

$T_{r+1}={}^6C_r(2x^2)^{6-r}\left(-\frac{3}{x^2}\right)^r$

$={}^6C_r\,2^{6-r}\,(-1)^r\times 3^r\times x^{12-2r-2r}$

Case-I :

For term independent of x is $12-4r=0\Rightarrow r=3$

$T_4=-\,{}^6C_3\times 2^3\times 3^3x^6=-20\times 2^3\times 3^3$

Case-II :

For term of x^{-8} $12-4r=-8\Rightarrow 4r=20\Rightarrow r=5$

$T_6={}^6C_5.\,2^1.\,(-1).\,3^5.\,x^{-8}$

Required ans. $=\frac{1}{60}\times(-20)2^3\times 3^3-\frac{1}{81}\times 6\times 2\times(-1)\times 3^5$

$=-72+36=-36$