

MATHEMATICS



* CLASS NOTES *
MATHEMATICS
Volume-1



CAREER POINT

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Volume-1

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Preface

Being involved in preparing students for competitive examination since 1993, we realized that students require books which should be self-sufficient, relevant and in student's friendly language like class notes. The book should develop sound understanding of fundamentals and also enhance questions solving ability of students. The book in your hand has been prepared to achieve these objectives.

We have developed this book using class notes of top faculty members of Career Point who have been successfully preparing students for JEE for more than two decade. Structure of book is such that you will feel like you are virtually attending the class of a Teacher. We firmly believe that the book in this form will definitely help a genuine, hardworking student to achieve target.

We have tried our best to keep errors out of this book. Comment and criticism from readers will be highly appreciated and incorporated in the subsequent edition.

We wish to utilize the opportunity to place on record our special thanks to all faculty members of Career Point and Content Development Team for their efforts to make this wonderful book.

Career Point Ltd.

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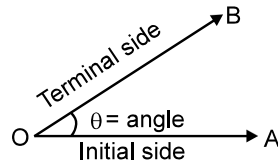
Trigonometrical Ratio

1. DEFINITION

Trigonometry is the branch of science in which we study about the angles and sides of a triangle.

1.1 Angle :

Consider a ray \vec{OA} . If this ray rotates about its end points O and takes the position OB, then the angle $\angle AOB$ has been generated.

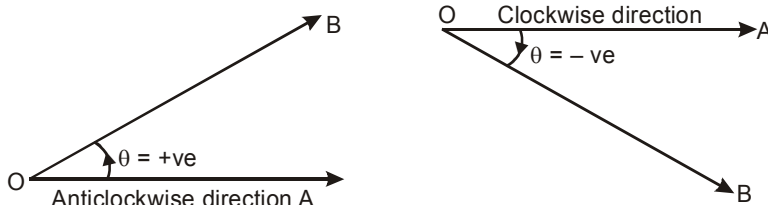


An angle is considered as the figure obtained by rotating a given ray about its end-point.

The initial position OA is called the initial side and the final position OB is called terminal side of the angle. The end point O about which the ray rotates is called the vertex of the angle.

1.2 Sense of an Angle :

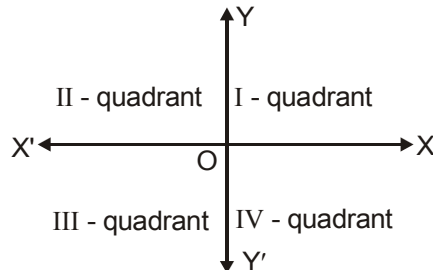
The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.



1.3 Some Useful terms :

1.3.1 Quadrant :

Let XOX' and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants.

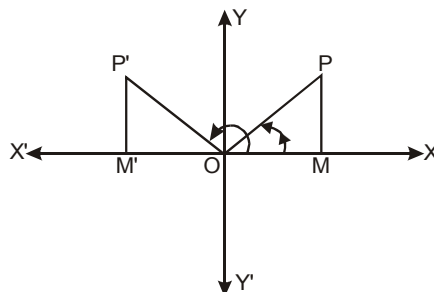


NOTES

The lines XOX' and YOY' are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY , YOX' , $X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrant respectively.

1.3.2 Angle In Standard Position : An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.

1.3.3 Co-terminal Angles : Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.



2. SYSTEM OF MEASUREMENT OF ANGLE

There are three system for measuring angles.

2.1 Sexagesimal system

2.2 Centesimal system

2.3 Circular System

2.1 Sexagesimal system :

The principal unit in this system is degree ($^{\circ}$). One right angle is divided into 90 equal parts and each part is called one degree (1°). One degree is divided into 60 equal parts and each part is called one minute. Minute is denoted by ($1'$). One minute is equally divided into 60 equal parts and each part is called one second ($1''$).

In Mathematical form :

One right angle = 90°

$1^{\circ} = 60'$

$1' = 60''$

➤ Examples Based On : SEXAGESIMAL SYSTEM

Ex.1 $45^{\circ} 15' 30''$ changes into degree

Sol. $60''$ is equal to $1'$

$1''$ is equal to $\left(\frac{1}{60}\right)'$

$30''$ is equal to $\left(\frac{1}{60} \times 30\right)' = \left(\frac{1}{2}\right)'$

Total minutes $\Rightarrow 15' + \left(\frac{1}{2}\right)' = \left(\frac{31}{2}\right)'$

$60'$ is equal to 1° and $1'$ is equal to $\left(\frac{1}{60}\right)^{\circ}$

NOTES

$$\left(\frac{31}{2}\right)' \text{ is equal to } \left(\frac{1}{60} \times \frac{31}{2}\right)^\circ = \left(\frac{31}{120}\right)^\circ$$

$$\text{Total degrees} \Rightarrow 45^\circ + \left(\frac{31}{120}\right)^\circ$$

$$\Rightarrow \left(\frac{5400 + 31}{120}\right)^\circ \Rightarrow \left(\frac{5431}{120}\right)^\circ$$

2.2 Centesimal system :

The principal unit in system is grade and is denoted by (^g). One right angle is divided into 100 equal parts, called grades, and each grade is subdivided into 100 minutes, and each minutes into 100 seconds.

In Mathematical Form :

$$\text{One right angle} = 100^g$$

$$1^g = 100'$$

$$1' = 100''$$

➤ *Examples Based On :* **CENTESIMAL SYSTEM**

Ex.2 50^g 30' 50'' change into grade system.

Sol. We know that , 50' $\Rightarrow \left(\frac{1}{2}\right)'$

$$\text{Total minute } 30' + \left(\frac{1}{2}\right)' = \left(\frac{61}{2}\right)'$$

100' is equal to 1^g

$$1' \text{ is equal to } \left(\frac{1}{100}\right)^g$$

$$\left(\frac{61}{2}\right)' \text{ is equal to } \left(\frac{1}{100} \times \frac{61}{2}\right)^g = \left(\frac{61}{200}\right)^g$$

$$\text{Total grade} \Rightarrow 50^g + \left(\frac{61}{200}\right)^g$$

$$\Rightarrow \left(\frac{10000 + 61}{200}\right)^g \Rightarrow \left(\frac{10061}{200}\right)^g$$

2.2.1 Relation between sexagesimal and centesimal systems :

One right angle = 90° (degree system) (1)

One right angle = 100^g (grade system) (2)

by (1) and (2),

$$90^\circ = 100^g$$

or,
$$\frac{D}{90} = \frac{G}{100}$$

then we can say,

$$1^\circ = \left(\frac{100}{90}\right)^g$$

$$1^g = \left(\frac{9}{10}\right)^\circ$$

NOTES

➤ **Examples Based On : RELATION BETWEEN SEXAGESIMAL AND CENTESIMAL SYSTEMS**

Ex.3 $63^\circ 14' 51''$ change into grade system.

Sol. We know that in degree system
60'' equal to 1'

$$51'' \text{ is equals } = \left(\frac{51}{60}\right)' = (0.85)'$$

(14.85)' change into degree.

$$(14.85)' \text{ is equals } = \left(\frac{14.85}{60}\right)^\circ = (0.2475)^\circ$$

So $63^\circ 14' 51'' = 63.2475^\circ$

63.2475° change into grade system.

$$63.2475^\circ \text{ is equals } = \left(63.2475 \times \frac{10}{9}\right)^g$$

$$= 70.2750^g$$

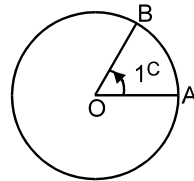
$$70.2750^g = 70^g 27' 50''$$

finally we can say,

$$63^\circ 14' 57'' = 70^g 27' 50''$$

2.3 Circular System :

One radian, written as 1^c , is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle. Consider a circle of radius r having centre at O . Let A be a point on the circle. Now cut off an arc AB whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOB$ is 1 radian (1^c).



2.3.1 Some Important Conversion :

$$\pi \text{ Radian} = 180^\circ$$

$$\text{One radian} = \left(\frac{180}{\pi}\right)^\circ$$

$$\frac{\pi}{6} \text{ Radian} = 30^\circ$$

$$\frac{\pi}{4} \text{ Radian} = 45^\circ$$

$$\frac{\pi}{3} \text{ Radian} = 60^\circ$$

$$\frac{\pi}{2} \text{ Radian} = 90^\circ$$

$$\frac{2\pi}{3} \text{ Radian} = 120^\circ$$

NOTES

$$\frac{3\pi}{4} \text{ Radian} = 135^\circ$$

$$\frac{5\pi}{6} \text{ Radian} = 150^\circ$$

$$\frac{7\pi}{6} \text{ Radian} = 210^\circ$$

$$\frac{5\pi}{4} \text{ Radian} = 225^\circ$$

$$\frac{5\pi}{3} \text{ Radian} = 300^\circ$$

2.3.2 Relation between systems of measurement of angles :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

➤ *Examples Based On :* **RELATION BETWEEN SYSTEMS OF MEASUREMENT OF ANGLES**

Ex.4 $\left(\frac{2\pi}{15}\right)^C$ change into degree system.

Sol. We know that, π radian = 180°

$$1^C = \left(\frac{180}{\pi}\right)^\circ$$

$$\left(\frac{2\pi}{15}\right)^C = \left(\frac{2\pi}{15} \times \frac{180}{\pi}\right)^\circ = 24^\circ$$

Ex.5 Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15° .

Sol. Let s be the length of the arc subtending an angle θ at the centre of a circle of radius r .

$$\text{then, } \theta = \frac{s}{r}$$

$$\text{Here, } r = 5 \text{ cm, and } \theta = 15^\circ = \left(15 \times \frac{\pi}{180}\right)^C$$

$$\theta = \left(\frac{\pi}{12}\right)^C$$

$$\theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5}$$

$$s = \frac{5\pi}{12} \text{ cm.}$$

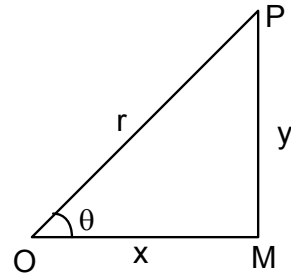
3. TRIGONOMETRICAL RATIOS OR FUNCTIONS

In the right angled triangle OMP, we have base (OM) = x , perpendicular (PM) = y and hypotenuse (OP) = r , then we define the following trigonometric ratios which are known as trigonometric function.

$$\sin\theta = \frac{P}{H} = \frac{y}{r}$$

NOTES

$$\begin{aligned} \cos\theta &= \frac{B}{H} = \frac{x}{r} \\ \tan\theta &= \frac{P}{B} = \frac{y}{x} \\ \cot\theta &= \frac{B}{P} = \frac{x}{y} \\ \sec\theta &= \frac{H}{B} = \frac{r}{x} \\ \operatorname{cosec}\theta &= \frac{H}{P} = \frac{r}{y} \end{aligned}$$



Note:

- (1) It should be noted that $\sin\theta$ does not mean the product of \sin and θ . The $\sin\theta$ is correctly read \sin of angle θ .
- (2) These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ .

3.1 Fundamental Trigonometrical Identities :

(a) $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$

(b) $\cos\theta = \frac{1}{\sec\theta}$

(c) $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

(d) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec^2\theta - \tan^2\theta = 1$

$$(\sec\theta - \tan\theta) = \frac{1}{(\sec\theta + \tan\theta)}$$

(e) $\sin^2\theta + \cos^2\theta = 1$

(f) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$(\operatorname{cosec}\theta - \cot\theta) = \frac{1}{\operatorname{cosec}\theta + \cot\theta}$$

➤ **Examples Based On : TRIGONOMETRICAL RATIOS OR FUNCTIONS**

Ex.6 Prove that, $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta \cos^2\theta)$

Sol. L.H.S, $(\sin^8\theta - \cos^8\theta)$
 or, $(\sin^4\theta)^2 - (\cos^4\theta)^2$
 or, $(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)[(\sin^4\theta + \cos^4\theta)]$
 or, $(\sin^2\theta - \cos^2\theta) [(\sin^2\theta + \cos^2\theta) - 2\sin^2\theta \cos^2\theta]$
 or, $(\sin^2\theta - \cos^2\theta) [(1 - 2\sin^2\theta \cos^2\theta)] = \text{RHS}$

Ex.7 Prove the identity $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$

Sol. L.H.S = $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$$

NOTES

$$\begin{aligned}
 [\because \sec^2\theta - \tan^2\theta &= 1] \\
 &= \frac{(\sec\theta + \tan\theta)\{1 - (\sec\theta - \tan\theta)\}}{\tan\theta - \sec\theta + 1} \\
 &= \frac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{\tan\theta - \sec\theta + 1} \\
 &= \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta} = \text{RHS}
 \end{aligned}$$

3.2 Signs of the trigonometrical ratios or functions :

Their signs depends on the quadrant in which the terminal side of the angle lies.

In First quadrant : $x > 0, y > 0 \Rightarrow \sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} > 0,$

$\tan\theta = \frac{y}{x} > 0, \operatorname{cosec}\theta = \frac{r}{y} > 0, \sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} > 0$

Thus, in the first quadrant all trigonometry functions are positive.

In Second quadrant : $x < 0, y > 0 \Rightarrow \sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} < 0,$

$\tan\theta = \frac{y}{x} < 0, \operatorname{cosec}\theta = \frac{r}{y} > 0, \sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} < 0$

Thus, in the second quadrant sin and cosec function are positive and all others are negative.

In Third quadrant : $x < 0, y < 0 \Rightarrow \sin\theta = \frac{y}{r} < 0, \cos\theta = \frac{x}{r} < 0,$

$\tan\theta = \frac{y}{x} > 0, \operatorname{cosec}\theta = \frac{r}{y} < 0, \sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} > 0$

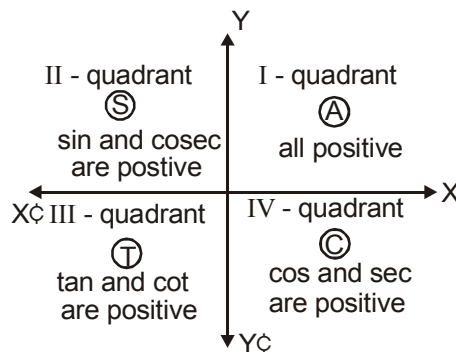
Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent.

In Fourth quadrant : $x > 0, y < 0 \Rightarrow \sin\theta = \frac{y}{r} < 0, \cos\theta = \frac{x}{r} > 0,$

$\tan\theta = \frac{y}{x} < 0, \operatorname{cosec}\theta = \frac{r}{y} < 0, \sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0$

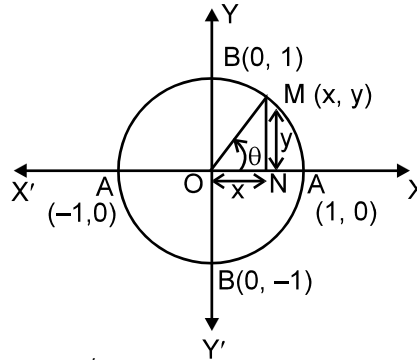
Thus, in the fourth quadrant all trigonometric functions are negative except cos and sec.

To be Remember :



A crude aid to memorise the signs of trigonometrical ratio in different quadrant.

NOTES

"All Students to Career Point"**3.3 Variations in values of Trigonometrical Functions in Different Quadrants :**

Let XOX' and YOY' be the coordinate axes. Draw a circle with centre at origin O and radius unity.

Let $M(x, y)$ be a point on the circle such that $\angle AOM = \theta$ then $x = \cos\theta$ and $y = \sin\theta$ – $1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$ for all values of θ .

I – Quadrant (A)

$\sin\theta$ \longrightarrow increases from 0 to 1
 $\cos\theta$ \longrightarrow decreases from 1 to 0
 $\tan\theta$ \longrightarrow increases from 0 to ∞
 $\cot\theta$ \longrightarrow decreases from ∞ to 0
 $\sec\theta$ \longrightarrow increases from 1 to ∞
 $\operatorname{cosec}\theta$ \longrightarrow decreases from ∞ to 1

II – Quadrant (S)

$\sin\theta$ \longrightarrow decreases from 1 to 0
 $\cos\theta$ \longrightarrow decreases from 0 to -1
 $\tan\theta$ \longrightarrow increases from $-\infty$ to 0
 $\cot\theta$ \longrightarrow decreases from 0 to $-\infty$
 $\sec\theta$ \longrightarrow increases from $-\infty$ to -1
 $\operatorname{cosec}\theta$ \longrightarrow increases from 1 to ∞

III – Quadrant (T)

$\sin\theta$ \longrightarrow decreases from 0 to -1
 $\cos\theta$ \longrightarrow increases from -1 to 0
 $\tan\theta$ \longrightarrow increases from 0 to ∞
 $\cot\theta$ \longrightarrow decreases from ∞ to 0
 $\sec\theta$ \longrightarrow decreases from -1 to $-\infty$
 $\operatorname{cosec}\theta$ \longrightarrow increases from $-\infty$ to -1

NOTES

IV – Quadrant (C)

$\sin\theta$	→ increases from -1 to 0
$\cos\theta$	→ increases from 0 to 1
$\tan\theta$	→ increases from $-\infty$ to 0
$\cot\theta$	→ decreases from 0 to $-\infty$
$\sec\theta$	→ decreases from ∞ to 1
$\operatorname{cosec}\theta$	→ decreases from -1 to $-\infty$

Remark:

$+\infty$ and $-\infty$ are two symbols. These are not real number. When we say that $\tan\theta$ increases from 0 to ∞ for as θ varies from 0 to $\frac{\pi}{2}$ it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains large positive values as θ tends to $\frac{\pi}{2}$. Similarly for other trigo. functions.

➤ **Examples Based On: SIGNS OF THE TRIGONOMETRICAL RATIOS OR FUNCTION**

Ex.8 If $\sec\theta = \sqrt{2}$, and $\frac{3\pi}{2} < \theta < 2\pi$. Find the value of $\frac{1 + \tan\theta + \operatorname{cosec}\theta}{1 + \cot\theta - \operatorname{cosec}\theta}$

Sol. If $\sec\theta = \sqrt{2}$

$$\text{or, } \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - \frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

But θ lies in the fourth quadrant in which $\sin\theta$ is negative.

$$\sin\theta = -\frac{1}{\sqrt{2}}, \operatorname{cosec}\theta = -\sqrt{2}$$

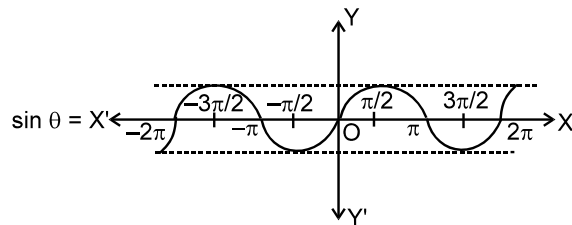
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1}$$

$$\Rightarrow \tan\theta = -1$$

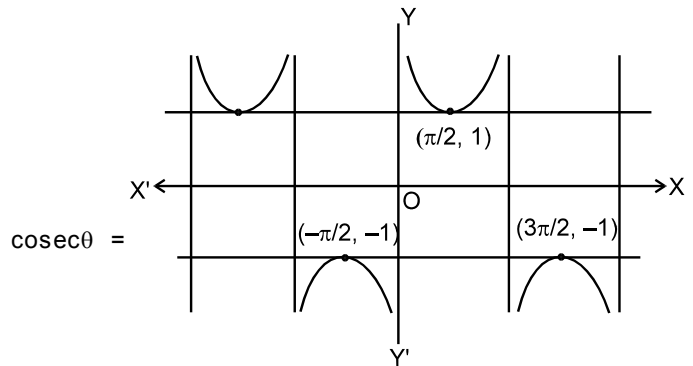
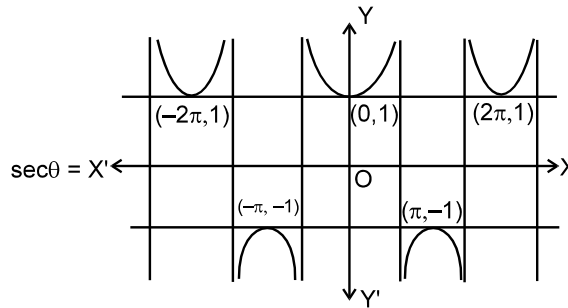
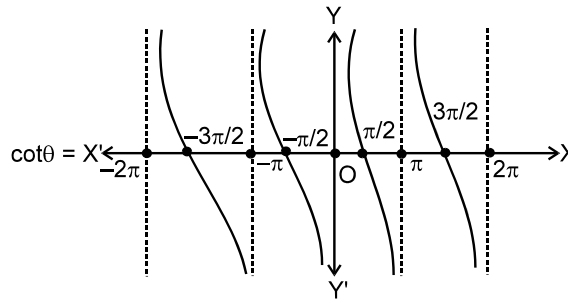
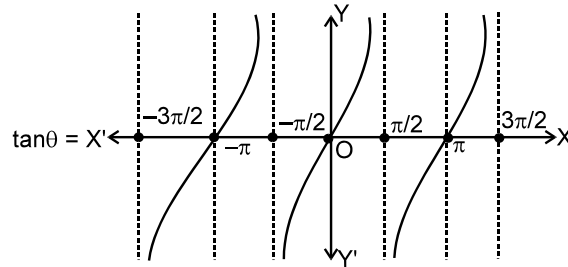
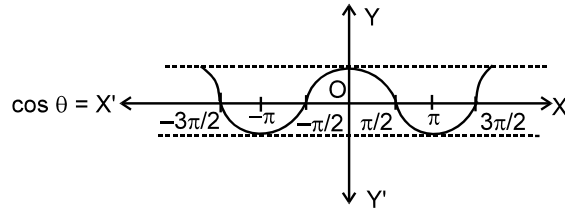
$$\Rightarrow \cot\theta = -1$$

$$\text{then, } \frac{1 + \tan\theta + \operatorname{cosec}\theta}{1 + \cot\theta - \operatorname{cosec}\theta} = \frac{1 - 1 - \sqrt{2}}{1 - 1 + \sqrt{2}} \Rightarrow -1$$

4. GRAPHS OF DIFFERENT TRIGONOMETRICAL RATIOS



NOTES



4.1 Domain and Range of Trigonometrical Function

Trig. Function	Domain	Range
$\sin \theta$	\mathbb{R}	$[-1, 1]$
$\cos \theta$	\mathbb{R}	$[-1, 1]$
$\tan \theta$	$\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$	$(-\infty, \infty)$ or \mathbb{R}
$\operatorname{cosec} \theta$	$\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec \theta$	$\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot \theta$	$\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$	$(-\infty, \infty) = \mathbb{R}$

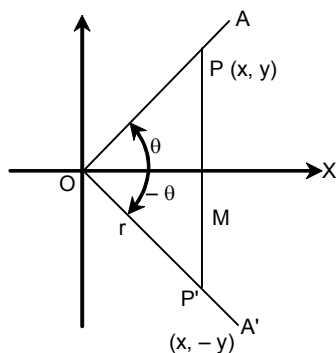
NOTES

5. TRIGONOMETRICAL RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90° .

5.1 Trigonometrical Ratios of $(-\theta)$:

Let a revolving ray starting from its initial position OX , trace out an angle $\angle XO A = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw $PM \perp$ from P on x -axis. angle $\angle XO A' = -\theta$ in the clockwise sense. Let P' be a point on OA' such that $OP' = OP$. Clearly M and M' coincide and $\triangle OMP$ is congruent to $\triangle OMP'$. then P' are $(x, -y)$



$$\sin(-\theta) = \frac{-y}{r} \Rightarrow \frac{-y}{r} = -\sin \theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos \theta$$

$$\tan(-\theta) = \frac{-y}{x} = -\tan \theta$$

NOTES

Taking the reciprocal of these trigonometric ratios,

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$$

$$\sec(-\theta) = \sec\theta \text{ and } \cot(-\theta) = \cot\theta$$

Note : A function $f(x)$ is said to be even function if $f(-x) = f(x)$ for all x in its domain.

A function $f(x)$ is an odd function if $f(-x) = -f(x)$ for all x in its domain.
 $\sin\theta$, $\tan\theta$, $\cot\theta$, $\operatorname{cosec}\theta$ all odd functions and $\cos\theta$, $\sec\theta$ are even functions.

➤ **Examples Based On : ALLIED ANGLES**

Ex.9 Find the value of the following trigonometric ratios -

(a) $\cos(-45^\circ)$ (b) $\sin(-30^\circ)$ (c) $\cot(-60^\circ)$

Sol. (a) $\cos(-45^\circ) = \cos 45^\circ$ [$\because \cos(-\theta) = \cos\theta$]

$$= \frac{1}{\sqrt{2}} \quad \text{Ans.}$$

(b) $\sin(-30^\circ) = -\sin 30^\circ$ [$\because \sin(-\theta) = -\sin\theta$]

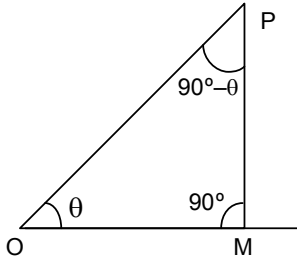
$$= -\frac{1}{2} \quad \text{Ans.}$$

(c) $\cot(-60^\circ) = -\cot 60^\circ$ [$\because \cot(-\theta) = -\cot\theta$]

$$= -\frac{1}{\sqrt{3}} \quad \text{Ans.}$$

5.2 Trigonometrical Functions of $(90 - \theta)$:

Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ . From any point P on it draw $PM \perp$ to OA. Three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is right angle.



$$\angle OPM = 90^\circ - \theta.$$

[When the angle OPM is considered, the line PM is the 'base' and MO is the 'perpendicular']

$$\sin(90^\circ - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos\theta$$

$$\cos(90^\circ - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin\theta$$

$$\tan(90^\circ - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta$$

$$\cot(90^\circ - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta$$

$$\operatorname{cosec}(90^\circ - \theta) = \operatorname{cosec} MPO = \frac{PO}{MO} = \sec AOP = \sec \theta$$

$$\text{and } \sec(90^\circ - \theta) = \sec MPO = \frac{PO}{PM} = \operatorname{cosec} AOP = \operatorname{cosec} \theta$$

Trigo. ratio	$(-\theta)$	$90 - \theta$	$90 + \theta$	$180 - \theta$	$180 + \theta$	$270 - \theta$	$270 + \theta$	$360 - \theta$
		or $\left(\frac{\pi}{2} - \theta\right)$	or $\left(\frac{\pi}{2} + \theta\right)$	or $(\pi - \theta)$	or $(\pi + \theta)$	or $\left(\frac{3\pi}{2} - \theta\right)$	or $\left(\frac{3\pi}{2} + \theta\right)$	or $(2\pi - \theta)$
$\sin \theta$	$-\sin \theta$	$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$
$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$	$\cos \theta$
$\tan \theta$	$-\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$

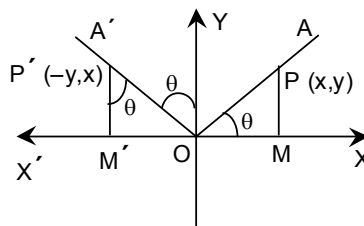
5.3 Trigonometrical Functions of $(90 + \theta)$:

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA' starting from the same initial position OX, first trace out an angle θ so as to coincide with OA and then it revolves through an angle of 90° in anticlockwise direction to form an angle $\angle XOA' = 90^\circ + \theta$.

Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$.

Draw perpendicular PM and P'M' from P and P' respectively on OX. Let the coordinates of P be (x, y) . Then $OM = x$ and $PM = y$ clearly,

$OM' = PM = y$ and $P'M' = OM = x$



so the coordinates of P' are $(-y, x)$

$$\sin(90 + \theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos \theta$$

$$\cos(90 + \theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin \theta$$

$$\tan(90 + \theta) = \frac{M'P'}{OM'} = -\frac{x}{y} = -\cot \theta$$

similarly,

$$\cot(90 + \theta) = -\tan \theta$$

$$\sec(90 + \theta) = -\operatorname{cosec} \theta$$

$$\operatorname{cosec}(90 + \theta) = \sec \theta$$

[where $-\pi/2 < \theta < \pi/2$]

NOTES

5.4 Periodic Function :

All the trigonometric functions are periodic functions. They will repeat after a certain period

$$\left. \begin{aligned} \sin(2n\pi + \theta) &= \sin\theta \\ \cos(2n\pi + \theta) &= \cos\theta \\ \tan(2n\pi + \theta) &= \tan\theta \end{aligned} \right\} \text{ where } n \in \mathbb{I}$$

➤ **Examples Based On : TRIGONOMETRIC RATIO OF ALLIED ANGLES**

Ex.10 Prove that , $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ = -1$

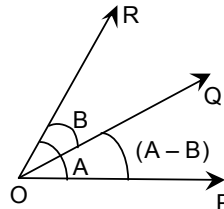
Sol. LHS = $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ$
 $= \cos(360^\circ + 150^\circ) \cos(360^\circ - 30^\circ) + \sin(360^\circ + 30^\circ) \cos(90^\circ + 30^\circ)$
 $= \cos 150^\circ \cos 30^\circ - \sin 30^\circ(-\sin 30^\circ)$
 $= \cos(180^\circ - 30^\circ) \frac{3}{4} + \frac{1}{4}$
 $= -\cos 30^\circ \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}$
 $= -\frac{3}{4} - \frac{1}{4} = -1 = \text{R.H.S}$

6. SUM OR DIFFERENCE OF THE ANGLES

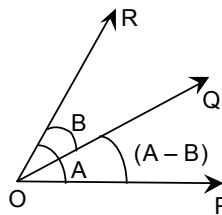
The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example : If A, B, C are three angles then $A \pm B$, $A + B + C$, $A - B + C$ etc. are compound angles.

6.1 (a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$



(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$



(c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

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$$(e) \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$(f) \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$(g) \cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$(h) \cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

6.2 Some More Results :

$$*(a) \sin(A + B) \cdot \sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$*(b) \cos(A + B) \cdot \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

$$(c) \sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \sin C \\ + \cos A \cos B \sin C - \sin A \sin B \sin C$$

$$(d) \cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C \\ - \sin A \cos B \sin C - \sin A \sin B \cos C$$

$$(e) \tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

(Not : * Important)➤ **Examples Based On : SUM OR DIFFERENCE OF THE ANGLE**

Ex.11 If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A < \frac{\pi}{2}$, $0 < B < \frac{\pi}{2}$, find the

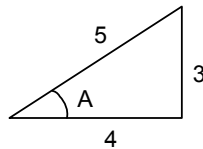
values of the following -

(a) $\sin(A + B)$ (b) $\cos(A - B)$

Sol. (a) $\sin(A + B) \Rightarrow \sin A \cos B + \cos A \sin B$

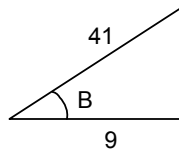
$$\sin A = \frac{3}{5}$$

$$\cos A = \frac{4}{5}$$



$$\text{and } \cos B = \frac{9}{41}$$

$$\sin B = \frac{40}{41}$$



$$\sin(A + B) = \frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$$

NOTES

$$(b) \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{4}{5} \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$$

7. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

We know that,

$$\sin A \cos B + \cos A \sin B = \sin(A + B) \dots\dots(i)$$

$$\sin A \cos B - \cos A \sin B = \sin(A - B) \dots\dots(ii)$$

$$\cos A \cos B - \sin A \sin B = \cos(A + B) \dots\dots(iii)$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B) \dots\dots(iv)$$

Adding (i) and (ii),

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

Subtracting (ii) from (i),

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

Adding (iii) and (iv),

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

Subtraction (iii) from (iv).

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

Formula:

$$(a) 2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$(b) 2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$(c) 2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$(d) 2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

➤ **Examples Based On : TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE**

Ex.12 Prove that,

$$\cos(30^\circ - A) \cdot \cos(30^\circ + A) + \cos(45^\circ + A) \cdot \cos(45^\circ - A) = \cos 2A + \frac{1}{4}$$

$$\text{Sol. L.H.S.} = \cos(30^\circ - A) \cdot \cos(30^\circ + A) + \cos(45^\circ + A) \cdot \cos(45^\circ - A)$$

$$= \frac{1}{2} [2 \cos(30^\circ - A) \cdot \cos(30^\circ + A) + 2 \cos(45^\circ + A) \cdot \cos(45^\circ - A)]$$

$$= \frac{1}{2} [\cos 60^\circ + \cos 2A + \cos 90^\circ + \cos 2A]$$

$$= \frac{1}{2} [2 \cos 2A + \frac{1}{2}]$$

$$= \cos 2A + \frac{1}{4} = \text{R.H.S.}$$

8. FORMULAE TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

We know that,

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \dots\dots(i)$$

Let $A + B = C$ and $A - B = D$

$$\text{then } A = \frac{C+D}{2} \text{ and } B = \frac{C-D}{2}$$

Substituting in (i),

$$(a) \sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cdot \cos\left(\frac{C-D}{2}\right)$$

similarly other formula,

$$(b) \sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right)$$

$$(c) \cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cdot \cos\left(\frac{C-D}{2}\right)$$

$$(d) \cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{D-C}{2}\right)$$

➤ **Examples Based On : TO TRANSFORM THE SUM OF DIFFERENCE INTO PRODUCT**

Ex.13 Prove that, $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

Sol. L.H.S,

$$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2$$

$$\left[2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)\right]^2 + \left[2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)\right]^2$$

$$= 4\cos^2\left(\frac{\alpha + \beta}{2}\right) \cdot \cos^2\left(\frac{\alpha - \beta}{2}\right) + 4\sin^2\left(\frac{\alpha + \beta}{2}\right) \cdot \cos^2\left(\frac{\alpha - \beta}{2}\right)$$

$$= 4 \cos^2\left(\frac{\alpha - \beta}{2}\right) \cdot \left[\cos^2\left(\frac{\alpha + \beta}{2}\right) + \sin^2\left(\frac{\alpha + \beta}{2}\right)\right] = 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$$

9. TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES

Trigonometric ratios of an angle 2A in terms of an angle A :

$$(a) \sin 2A = 2\sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$(b) \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1$$

$$= 1 - 2 \sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$(c) \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$(d) \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$(e) \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$(f) \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

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(g) $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

(h) $\tan A = \frac{1 - \cos 2A}{\sin 2A}$

(i) $\sqrt{1 + \sin 2A} = |\sin A + \cos A|$

(j) $\sqrt{1 - \sin 2A} = |\sin A - \cos A|$

➤ Examples Based On : TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES

Ex.14 Prove that , $\frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \left(\frac{\theta}{2} \right)$

Sol. L.H.S = $\frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \frac{(1 - \cos \theta) + \sin \theta}{(1 + \cos \theta) + \sin \theta}$

$$= \frac{2 \sin^2 \left(\frac{\theta}{2} \right) + 2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)}{2 \cos^2 \left(\frac{\theta}{2} \right) + 2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)}$$

$$= \frac{2 \sin \left(\frac{\theta}{2} \right) \left[\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right]}{2 \cos \left(\frac{\theta}{2} \right) \left[\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right]} = \tan \left(\frac{\theta}{2} \right)$$

$$= \text{R.H.S}$$

Ex.15 Show that, $\sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos 8\theta}}} = 2 \cos \theta$ where $\theta \in \left[-\frac{\pi}{16}, \frac{\pi}{16} \right]$

Sol. L.H.S.,

$$= \sqrt{2 + \sqrt{2 + \sqrt{2 + 2 \cos 8\theta}}} \left[1 + \cos 8\theta = 2 \cos^2 \left(\frac{8\theta}{2} \right) \right]$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{2(2 \cos^2 4\theta)}}}$$

$$= \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}} = \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}}$$

$$= \sqrt{2 + \sqrt{2(2 \cos^2 2\theta)}} = \sqrt{2 + 2 \cos 2\theta}$$

$$= \sqrt{2(1 + \cos 2\theta)} = \sqrt{2(2 \cos^2 \theta)}$$

$$= 2 \cos \theta = \text{R.H.S}$$

10. CONDITIONAL TRIGONOMETRICAL IDENTITIES

We have certain trigonometric identities

like, $\sin^2 \theta + \cos^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If A, B, C denote the angle of a triangle ABC, then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratios of these angles.

(I) If $A + B + C = \pi$, then $A + B = \pi - C$, $B + C = \pi - A$ and $C + A = \pi - B$

(II) If $A + B + C = \pi$, then $\sin(A + B) = \sin(\pi - C) = \sin C$

similarly, $\sin(B + C) = \sin(\pi - A) = \sin A$

and $\sin(C + A) = \sin(\pi - B) = \sin B$

(III) If $A + B + C = \pi$, then $\cos(A + B) = \cos(\pi - C) = -\cos C$

similarly, $\cos(B + C) = \cos(\pi - A) = -\cos A$

and $\cos(C + A) = \cos(\pi - B) = -\cos B$

(IV) If $A + B + C = \pi$, then $\tan(A + B) = \tan(\pi - C) = -\tan C$

similarly, $\tan(B + C) = \tan(\pi - A) = -\tan A$

and, $\tan(C + A) = \tan(\pi - B) = -\tan B$

(V) If $A + B + C = \pi$, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ and $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$

and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$$

$$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)$$

$$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$$

All problems on conditional identities are broadly divided into the following four types :

- (I) Identities involving sines and cosines of the multiple or sub-multiples of the angles involved.
- (II) Identities involving squares of sines and cosines of the multiple or sub-multiples of the angles involved.
- (III) Identities involving tangents and cotangents of the multiples or sub-multiples of the angles involved.
- (IV) Identities involving cubes and higher powers of sines and cosines and some mixed identities.

10.1 TYPE I : Identities involving sines and cosines of the multiple or sub-multiple of the angles involved.

Working Methods :

Step – 1 Express of the sum of first two terms as product by using C & D formulae.

Step – 2 In the product obtained in step II replace the sum of two angles in terms of the third by using the given relation.

Step – 3 Expand the third term by using formulae (Double angle change into single angle or change into half angle).

Step – 4 Taking common factor.

Step – 5 Express the trigonometric ratio of the single angle in terms of the remaining angles.

Step – 6 Use the one of the formulae given in the step I to convert the sum into product.

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➤ **Examples Based On : CONDITIONAL TRIGONOMETRICAL IDENTITIES TYPE I**

Ex.16 If $A + B + C = \pi$, prove that,

$$\cos A + \cos B + \cos C = 1 + 4 \sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right)$$

Sol. L.H.S. = $\cos A + \cos B + \cos C$

$$= 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2 \cos\left(\frac{\pi - C}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) + \cos C$$

$$= 2 \sin\left(\frac{C}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right)$$

$$= 2 \sin\left(\frac{C}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) - 2 \sin^2\left(\frac{C}{2}\right) + 1$$

$$= 2 \sin\left(\frac{C}{2}\right) \left[\cos\left(\frac{A - B}{2}\right) - \sin\left(\frac{C}{2}\right) \right] + 1$$

$$= 2 \sin\left(\frac{C}{2}\right) \left[\cos\left(\frac{A - B}{2}\right) - \sin\left(\frac{\pi - A + B}{2}\right) \right] + 1$$

$$= 2 \sin\left(\frac{C}{2}\right) \left[\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right) \right] + 1$$

$$= 2 \sin\left(\frac{C}{2}\right) \left[2 \sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \right] + 1$$

$$= 1 + 4 \sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \sin\left(\frac{C}{2}\right) = \text{R.H.S.}$$

Ex.17 If $A + B + C = \pi$, Prove that

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)$$

$$= 1 + 4 \sin\left(\frac{\pi - A}{4}\right) \cdot \sin\left(\frac{\pi - B}{4}\right) \cdot \sin\left(\frac{\pi - C}{4}\right)$$

$$= 1 + 4 \sin\left(\frac{B+C}{4}\right) \cdot \sin\left(\frac{C+A}{4}\right) \cdot \sin\left(\frac{A+B}{4}\right)$$

Sol. L.H.S. = $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)$

$$= 2 \sin\left(\frac{A+B}{4}\right) \cdot \cos\left(\frac{A-B}{4}\right) + \cos\left(\frac{\pi - C}{2}\right)$$

$$= 2 \sin\left(\frac{\pi - C}{4}\right) \cdot \cos\left(\frac{A - B}{4}\right) + 1 - 2 \sin^2\left(\frac{\pi - C}{4}\right)$$

$$= 2 \sin\left(\frac{\pi - C}{4}\right) \left[\cos\left(\frac{A - B}{4}\right) - \sin\left(\frac{\pi - C}{4}\right) \right] + 1$$

NOTES

$$\begin{aligned}
&= 2 \sin\left(\frac{\pi-C}{4}\right) \left[\cos\left(\frac{A-B}{4}\right) - \cos\left\{\frac{\pi}{2} - \left(\frac{\pi-C}{4}\right)\right\} \right] + 1 \\
&= 2 \sin\left(\frac{\pi-C}{4}\right) \left[\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right) \right] + 1 \\
&= 2 \sin\left(\frac{\pi-C}{4}\right) \\
&\left[2 \sin\left(\frac{A-B+\pi+C}{8}\right) \sin\left(\frac{\pi+C-A+B}{8}\right) \right] + 1 \\
&= 2 \sin\left(\frac{\pi-C}{4}\right) \\
&\left[2 \sin\left(\frac{A+C+\pi-B}{8}\right) \sin\left(\frac{\pi+C-A+B}{8}\right) \right] + 1 \\
&= 2 \sin\left(\frac{\pi-C}{4}\right) \\
&\left[2 \sin\left(\frac{\pi+B+\pi-B}{8}\right) \sin\left(\frac{\pi-A+\pi-A}{8}\right) \right] + 1 \\
&= 2 \sin\left(\frac{\pi-C}{4}\right) \left[2 \sin\left(\frac{\pi-B}{4}\right) \cdot \sin\left(\frac{\pi-A}{4}\right) \right] + 1 \\
&= 1 + 4 \sin\left(\frac{\pi-A}{4}\right) \cdot \sin\left(\frac{\pi-B}{4}\right) \cdot \sin\left(\frac{\pi-A}{4}\right) \\
&= 1 + 4 \sin\left(\frac{B+C}{4}\right) \cdot \sin\left(\frac{C+A}{4}\right) \cdot \sin\left(\frac{A+B}{4}\right) = \text{R.H.S}
\end{aligned}$$

10.2 TYPE II : Identities involving squares of sines and cosines of multiple or sub-multiples of the angles involved.

Working step :

- (I) Arrange the terms on the L.H.S of the identity so that either $\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$ or $\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B)$ can be used.
- (II) Take the common factor outside.
- (III) Express the trigonometric ratio of a single angle inside the bracket into that of the sum of the angles.
- (IV) Use the formulae to convert the sum into product.

➤ Examples Based On : **CONDITIONAL TRIGONOMETRICAL IDENTITIES** TYPE II

Ex.18 If $A + B + C = \pi$

Prove that, $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

Sol. I Method

$$\begin{aligned}
\text{L.H.S. } &\cos^2 A + \cos^2 B + \cos^2 C \\
&= \cos^2 A + (1 - \sin^2 B) + \cos^2 C \\
&= (\cos^2 A - \sin^2 B) + \cos^2 C + 1 \\
&\quad [\because A + B = \pi - C, \cos(A+B) = -\cos C] \\
&= \cos(A+B) \cdot \cos(A-B) + \cos^2 C + 1
\end{aligned}$$

NOTES

$$\begin{aligned}
 &= -\cos C \cdot \cos(A - B) + \cos^2 C + 1 \\
 &= -\cos C [\cos(A - B) - \cos C] + 1 \\
 &= -\cos C [\cos(A - B) + \cos(A + B)] + 1 \\
 &\qquad\qquad\qquad [\because \cos C = -\cos(A + B)] \\
 &= -\cos C [2 \cos A \cos B] + 1 \\
 &= 1 - 2 \cos A \cos B \cos C = \text{R.H.S.}
 \end{aligned}$$

II Method

$$\begin{aligned}
 &\cos^2 A + \cos^2 B + \cos^2 C \\
 &= \frac{1}{2} [2 \cos^2 A + 2 \cos^2 B + 2 \cos^2 C] \quad [\because \cos 2A = 2\cos^2 A - 1] \\
 &= \frac{1}{2} [(1 + \cos 2A) + (1 + \cos 2B) + (1 + \cos 2C)] \\
 &= \frac{1}{2} [3 + \cos 2A + \cos 2B + \cos 2C] \\
 &= \frac{3}{2} + \frac{1}{2} [\cos 2A + \cos 2B + \cos 2C] \\
 &= \frac{3}{2} + \frac{1}{2} [2 \cos(A + B) \cdot \cos(A - B) + 2\cos^2 C - 1] \\
 &= \frac{3}{2} + \frac{1}{2} [-2 \cos C \cos(A - B) + 2\cos^2 C - 1] \\
 &= \frac{3}{2} - \frac{1}{2} + \frac{1}{2} [-2\cos C \{\cos(A - B) - \cos C\}] \\
 &= 1 - \cos C [\cos(A - B) - \cos C] \\
 &= 1 - \cos C [\cos(A - B) + \cos(A + B)] \\
 &\qquad\qquad\qquad [\cos C = -\cos(A + B)] \\
 &= 1 - \cos C [2 \cos A \cdot \cos B] \\
 &= 1 - 2 \cos A \cdot \cos B \cos C
 \end{aligned}$$

10.3 Type III : Identities for tan and cot of the angles**Working step :**

- (I) Express the sum of the two angles in terms of third angle by using the given relation.
- (II) Taking tan from both the sides.
- (III) Expand the L.H.S in step II by using the formula for the tangent of the compound angles.
- (IV) Use cross multiplication in the expression obtained in the step III.
- (V) Arrange the terms as per the requirement in the sum.

➤ *Examples Based On* : **CONDITIONAL TRIGONOMETRICAL IDENTITIES**
TYPE III

Ex.19 If $x + y + z = xyz$

Prove that,
$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Sol. Let $x = \tan A$, $y = \tan B$, $z = \tan C$

NOTES

then $x + y + z = xyz$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$$

Dividing by $[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$ both the sides

$$\Rightarrow \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = 0$$

$$\Rightarrow \tan(A + B + C) = 0$$

$$\Rightarrow A + B + C = n\pi \quad [n \in \mathbb{Z}]$$

$$\text{Now, } A + B + C = n\pi$$

$$2A + 2B + 2C = 2n\pi$$

$$\Rightarrow \tan(2A + 2B + 2C) = \tan 2n\pi$$

$$\Rightarrow \frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C}{1 - \tan 2A \tan 2B - \tan 2B \tan 2C - \tan 2C \tan 2A} = 0$$

$$\Rightarrow \tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C = 0$$

$$\Rightarrow \tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C$$

$$\Rightarrow \frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C}$$

$$= \frac{2 \tan A}{1 - \tan^2 A} \cdot \frac{2 \tan B}{1 - \tan^2 B} \cdot \frac{2 \tan C}{1 - \tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2}$$

$$= \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

11. TO FIND THE GREATEST AND LEAST VALUE OF THE EXPRESSION

$[a \sin \theta + b \cos \theta]$

Let $a = r \cos \alpha$ (1)

and $b = r \sin \alpha$ (2)

Squaring and adding (1) and (2)

then $a^2 + b^2 = r^2$

or, $r = \sqrt{a^2 + b^2}$

$\therefore a \sin \theta + b \cos \theta$

$$= r (\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$= r \sin(\theta + \alpha)$$

But $-1 \leq \sin \theta \leq 1$

so $-1 \leq \sin(\theta + \alpha) \leq 1$

then $-r \leq r \sin(\theta + \alpha) \leq r$

NOTES

Hence, $-\sqrt{a^2+b^2} \leq a \sin\theta + b \cos\theta \leq \sqrt{a^2+b^2}$

then the greatest and least values of $a \sin\theta + b \cos\theta$ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

➤ **Examples Based On : TO FIND THE GREATEST AND LEAST VALUE OF THE EXPRESSION**

Ex.20 Prove that $5 \cos\theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between -4 and 10 .

Sol. The given expression is,

$$\begin{aligned} & 5 \cos\theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3 \\ \Rightarrow & 5 \cos\theta + 3[\cos\theta \cos 60^\circ - \sin\theta \sin 60^\circ] + 3 \\ \Rightarrow & 5 \cos\theta + 3\left[\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right] + 3 \\ \Rightarrow & \frac{1}{2}[13 \cos\theta - 3\sqrt{3} \sin\theta] + 3 \end{aligned}$$

$$\text{Put } 13 = r \cos\alpha, \quad 3\sqrt{3} = r \sin\alpha$$

$$r = \sqrt{169 + 27} = 14$$

$$\Rightarrow \frac{1}{2}[r \cos(\theta + \alpha)] + 3$$

$$\Rightarrow \frac{14}{2}[\cos(\theta + \alpha)] + 3$$

$$\Rightarrow 7 \cos(\theta + \alpha) + 3$$

Hence maximum and minimum values of expression are $(7+3)$ and $(-7+3)$ i.e., 10 and -4 respectively.

12. MISCELLANEOUS POINTS

(1) Some useful Identities :

$$(a) \tan(A + B + C) = \frac{\sum \tan A - \tan A \tan B \tan C}{1 - \sum \tan A \tan B}$$

$$(b) \cot\theta - \tan\theta = 2 \cot 2\theta$$

$$(c) \frac{1}{4} \sin 3\theta = \sin\theta \cdot \sin(60 - \theta) \cdot \sin(60 + \theta)$$

$$(d) \frac{1}{4} \cos 3\theta = \cos\theta \cdot \cos(60 - \theta) \cdot \cos(60 + \theta)$$

$$(e) \tan 3\theta = \tan\theta \cdot \tan(60 - \theta) \cdot \tan(60 + \theta)$$

$$(f) \tan(A + B) - \tan A - \tan B = \tan A \cdot \tan B \cdot \tan(A + B)$$

(2) Some useful result :

$$(a) \text{ver } \sin\theta = 1 - \cos\theta$$

$$(b) \text{cover } \sin\theta = 1 - \sin\theta$$

NOTES

(3) Some useful series :(a) $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots +$ to n terms

$$= \frac{\sin \left[\alpha + \left(\frac{n-1}{2} \right) \beta \right] \left[\sin \left(\frac{n\beta}{2} \right) \right]}{\sin \left(\frac{\beta}{2} \right)} ; \beta \neq 2n\pi$$

(b) $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots +$ to n terms

$$= \frac{\cos \left[\alpha + \left(\frac{n-1}{2} \right) \beta \right] \left[\sin \left(\frac{n\beta}{2} \right) \right]}{\sin \left(\frac{\beta}{2} \right)} ; \beta \neq 2n\pi$$

➤ **Examples Based On : SERIES****Ex.21** Prove that $\cos \left(\frac{\pi}{14} \right) + \cos \left(\frac{3\pi}{14} \right) + \cos \left(\frac{5\pi}{14} \right) = \frac{1}{2} \cot \left(\frac{\pi}{14} \right)$ **Sol.** Here $\alpha = \frac{\pi}{14}$, $\beta = \frac{2\pi}{14}$ and $n = 3$.

$$S = \frac{\cos \left[\frac{\pi}{14} + \left(\frac{3-1}{2} \right) \left(\frac{2\pi}{14} \right) \right] \sin \left(\frac{2\pi}{14} \times \frac{3}{2} \right)}{\sin \left(\frac{2\pi}{14} \times \frac{1}{2} \right)}$$

$$S = \frac{2 \cos \left(\frac{3\pi}{14} \right) \sin \left(\frac{3\pi}{14} \right)}{2 \sin \left(\frac{\pi}{14} \right)}$$

$$S = \frac{\sin \left(\frac{6\pi}{14} \right)}{2 \sin \left(\frac{\pi}{14} \right)} = \frac{1}{2} \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{14} \right)}{\sin \left(\frac{\pi}{14} \right)}$$

$$S = \frac{1}{2} \cot \left(\frac{\pi}{14} \right)$$

(4) An Increasing Product series :(a) $p = \cos \alpha \cdot \cos 2\alpha \cdot \cos 2^2\alpha \dots \cos (2^{n-1} \alpha)$

$$\begin{cases} \frac{\sin 2^n \alpha}{2^n \sin \alpha}, & \text{if } \alpha \neq n\pi \\ 1, & \text{if } \alpha = 2k\pi \\ -1, & \text{if } \alpha = (2k+1)\pi \end{cases}$$

NOTES

(5) sine, cosine and tangent of some angle less than 90°.

	15°	18°	22½°	36°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

(6) Conversion 1 radian = 180°/π = 57° 17' 45" (approximately)

and 1° = $\frac{\pi}{180}$ = 0.01745 radians (approximately)

(7) Basic right angled triangle are (pythogerian Triplets)

3, 4, 5; 5, 12, 13; 7, 24, 25; 8, 15, 17; 9, 40, 41; 11, 60, 61;
12, 35, 37; 20, 21, 29 etc.

(8) Each interior angle of a regular polygon of n sides

$$= \frac{n-2}{n} \times 180 \text{ degrees}$$